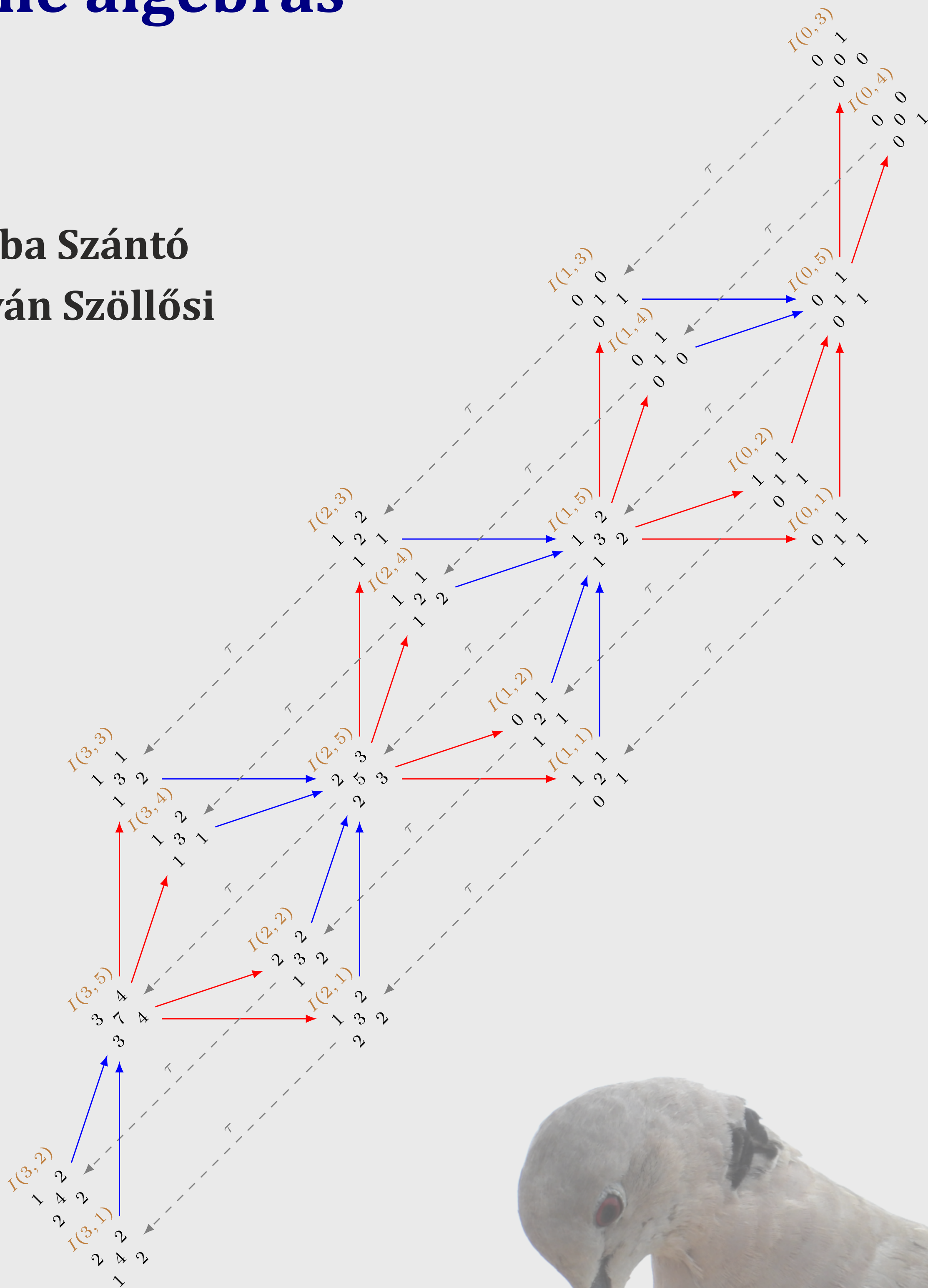
Combinatorial methods in the representation theory of finite dimensional tame algebras



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Introduction

Starting from classical problems in linear algebra and experiencing a spectacular development in the last forty years, the theory of representations of finite dimensional algebras is today one of the most popular and active fields in algebra. New theoretical approaches (like the use of quiver-theoretical techniques, the introduction of almost split sequences and Auslander-Reiten theory, the development of tilting theory, the notion of Gabriel-Roiter measure, the theory of Ringel-Hall algebras, the results in cluster theory) and the development of new computational algebra packages and tools (like GAP and Magma) have contributed not only to the deep understanding of the structure of module categories over some finite dimensional algebras, but have also connected representation theory with many other fields even outside algebra (for example quantum groups, algebraic geometry, control theory of linear dynamical systems).

We know that every basic, connected and hereditary finite dimensional algebra is the path algebra of an acyclic, connected quiver. Also, the category of modules over a path algebra is equivalent with the category of representations of the corresponding quiver. So modules in these cases can be identified with quiver representations.

In the present book we will be mainly interested in the category of representations of tame (affine, Euclidean) quivers over a finite field (i.e. the category of finite dimensional modules over the path algebra kQ, where Q is of type $\widetilde{\mathbb{A}}_m$ ($m \geq 1$), $\widetilde{\mathbb{D}}_m$ ($m \geq 4$), $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ and k is finite). Note that kQ with k finite is a finitary ring (i.e. the group of extensions of modules is finite). Our aim is to count certain monomorphisms, epimorphisms, automorphisms and extensions mainly for indecomposables. For this we will need many tools, but of special importance is the so-called Schofield induction via orthogonal exceptional pairs.

Knowing the number of extensions leads us to Ringel-Hall algebras with a large spectrum of applications. More precisely, the structure coefficients of the Ringel-Hall algebra associated to kQ – called Ringel-Hall numbers – are (up to automorphisms) the numbers of extensions with given middle terms.

Classical Hall algebras associated with discrete valuation rings were introduced by Steinitz and Hall to provide an algebraic approach to the classical combinatorics of partitions. The multiplication is given by Hall polynomials which play an important role in the representation theory of the symmetric groups and the general linear groups. In 1990 Ringel defined Hall algebras for a large class of rings, namely finitary rings, including in particular path algebras of quivers over finite fields. Far reaching analogues of the classical ones, these Ringel-Hall algebras provided a new approach to the study of quantum groups using the representation theory of finite dimensional algebras (see [41, 24]). They can also be used successfully in the theory of cluster algebras (see [10, 46]), to investigate the structure of the module category via the Gabriel-Roiter measure (see [45]) or to determine the cardinalities of some quiver Grassmannians over a finite field (see [46, 55]).

In case of Ringel-Hall algebras corresponding to Dynkin quivers and tame quivers we know due to Ringel, Hubery, respectively Deng and Ruan (see [42], [26], respectively [17]), that the structure coefficients of the multiplication (the Ringel-Hall numbers) are again polynomials in the number of elements of the base field. We will call these polynomials Ringel-Hall polynomials. If we are looking at Hall polynomials associated to indecomposable modules, the classical polynomials are just 0 or 1, and the Ringel-Hall polynomials in the Dynkin case are also known and have degree up to 5. However we do not have too much information about the Ringel-Hall polynomials in the tame case.

The present monograph records the progress made by the authors in the last fifteen years regarding tame Ringel-Hall polynomials and their various applications, using a large spectrum of combinatorial, computational and representation theoretical tools.

The first chapter is a preliminary one, serving as a comprehensive survey of the main notions and tools used throughout the monograph. It covers Gaussian coefficients, partition combinatorics, representations of tame quivers, reflection functors, orthogonal exceptional pairs and Schofield sequences, basic information on Ringel-Hall algebras and on the Gabriel-Roiter measure, as well as some geometrical connections.

The second chapter is dedicated to tame Hall polynomials. We describe all the tame Ringel-Hall products and polynomials involving indecomposables of absolute defect not higher than 1. These results were published in [54, 53]. Then we obtain some special Ringel-Hall polynomials of the form $F^{\delta}_{\delta-xx}$ (where x is a positive real root of arbitrary negative defect and δ is the minimal radical vector). This result appeared in [57]. Finally we deduce all the Ringel-Hall polynomials involving indecomposables of absolute defect up to 2, presenting the used combinatorial and representation theoretical framework, which can be generalized to indecomposables of arbitrary defect. This was published in [56, 58, 59]. Note that these results give us the full list of Ringel-Hall polynomials in the cases $\widetilde{\mathbb{A}}_m$ and $\widetilde{\mathbb{D}}_m$.

In the third chapter we will apply our knowledge on Ringel-Hall polynomials in the theory of Gabriel-Roiter measures. The Gabriel-Roiter measure (GR measure for short) was introduced by Gabriel in order to give a combinatorial interpretation of the induction scheme used by Roiter in his proof of the first Brauer-Thrall conjecture. Ringel used it as a foundational tool for the representation theory of Artin algebras. First of all we will prove that the GR inclusions in preprojective indecomposables and homogeneous modules of dimension δ, as well as their GR measures are field independent. A similar result for Dynkin quivers was obtained by Ringel in [45]. As an application of the respective theorems, we will prove (using Ringel-Hall polynomials) a result by Bo Chen from [15] in a more general context: our result is valid also for the case $\widetilde{\mathbb{E}}_8$ (this case is missing from [15]) and it is field independent (in [15] k is algebraically closed). More precisely, we prove that a GR submodule P of a homogeneous regular module R of dimension δ has defect -1. All these results appeared in [57].

In the fourth chapter we determine cardinalities of Kronecker quiver Grassmannians via Ringel-Hall numbers. We consider Grassmannian varieties of fixed dimensional submodules of indecomposable Kronecker modules. In [12] Caldero and Zelevinsky described (using Schubert cells) explicit combinatorial formulas for the Euler-Poincaré characteristics of these Grassmannians, using them in cluster theory. Using the Ringel-Hall algebra approach and reflection functors we obtain specific recursions for the Grassmannian cardinalities. All these, together with a q-analogue of a combinatorial identity due to Nanjundiah (presented in the preliminaries) give us explicit combinatorial formulas for the cardinalities of the mentioned Grassmannians. We realize in this way a quantification of the formulas by Caldero and Zelevinsky, with applications in quantum cluster theory. The results of this last chapter were published in [55].

Acknowledgments

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Chapter 1

Preliminaries

This chapter is dedicated to the introduction of the main notions and tools used throughout the monograph. For further details and clarifications we refer to [2, 49, 3, 16, 18, 19, 30, 43, 22, 32].

1.1 Gaussian coefficients

For $\ell, a \in \mathbb{Z}$, $\ell > 0$ we will denote by $\binom{a}{\ell}_q = \frac{(q^a-1)\dots(q^{a-\ell+1}-1)}{(q^\ell-1)\dots(q-1)}$ the Gaussian (q-binomial) coefficients. By definition $\binom{a}{0}_q = 1$ and $\binom{a}{-\ell}_q = 0$. Notice that for $a, \ell \geq 0$ the coefficient $\binom{a}{\ell}_q$ is a polynomial with integral coefficients in q, which expresses the number of points of the Grassmannian $Gr_\ell(k^a)$, where k is a finite field with q elements. As we will see in Remark 4.1 (b) we need the given definition of $\binom{a}{\ell}_q$ also in the case when $a < 0$.

The following properties of the Gaussian coefficients are well known:

Lemma 1.1. *We have the following:*

(a) $\binom{a}{\ell}_q = 0$, *for* $0 \leq a < \ell$. *Also* $\binom{a}{\ell}_q = \binom{a}{a-\ell}_q$, *for* $a, \ell \geq 0$;

(b) *(Cross product)* $\binom{a}{\ell}_q \binom{\ell}{j}_q = \binom{a}{j}_q \binom{a-j}{\ell-j}_q$, *for all* $a, \ell, j \in \mathbb{Z}$;

(c) *(q-Vandermonde)*

$$\binom{a+b}{\ell}_q = \sum_{j\in\mathbb{Z}} q^{j(a-\ell+j)} \binom{a}{\ell-j}_q \binom{b}{j}_q = \sum_{r\in\mathbb{Z}} q^{(\ell-r)(a-r)} \binom{a}{r}_q \binom{b}{\ell-r}_q,$$

for all $\ell, a, b \in \mathbb{Z}$. *Note that the sums are finite.*

Finally, we will prove a q-analogue of the so called Nanjundiah's identity (see [33]):

Proposition 1.1. *For all* $m, p, \mu, \nu \in \mathbb{Z}$ *we have the following:*

$$\sum_{r\in\mathbb{Z}} q^{(m-\mu+\nu-r)(p-r)} \binom{m-\mu+\nu}{r}_q \binom{p+\mu-\nu}{p-r}_q \binom{\mu+r}{m+p}_q = \binom{\mu}{m}_q \binom{\nu}{p}_q.$$

Proof. Denote by A the left expression and by B the right one.

One can immediately see that for $p < 0$ we have $A = B = 0$.

Applying three times Lemma 1.1 (c) and two times Lemma 1.1 (b) we have

$$\begin{aligned}
A &= \sum_{r\in\mathbb{Z}} q^{(m-\mu+\nu-r)(p-r)} \binom{m-\mu+\nu}{r}_q \binom{p+\mu-\nu}{p-r}_q \binom{\mu+r}{m+p}_q \\
&= \sum_{r} q^{(m-\mu+\nu-r)(p-r)} \binom{m-\mu+\nu}{r}_q \binom{p+\mu-\nu}{p-r}_q \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{\mu}{m+p-s}_q \binom{r}{s}_q \\
&= \sum_{r,s\in\mathbb{Z}} q^{s(\mu-m-p+s)} q^{(m-\mu+\nu-r)(p-r)} \binom{m-\mu+\nu}{r}_q \binom{r}{s}_q \binom{p+\mu-\nu}{p-r}_q \binom{\mu}{m+p-s}_q \\
&= \sum_{r,s\in\mathbb{Z}} q^{s(\mu-m-p+s)} q^{(m-\mu+\nu-r)(p-r)} \binom{m-\mu+\nu}{s}_q \binom{m-\mu+\nu-s}{r-s}_q \binom{p+\mu-\nu}{p-r}_q \binom{\mu}{m+p-s}_q \\
&= \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{m+p-s}_q \sum_{r\in\mathbb{Z}} q^{(m-\mu+\nu-r)(p-r)} \binom{m-\mu+\nu-s}{r-s}_q \binom{p+\mu-\nu}{p-r}_q \\
&= \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{m+p-s}_q \sum_{t\in\mathbb{Z}} q^{(m-\mu+\nu-s-t)(p-s-t)} \binom{m-\mu+\nu-s}{t}_q \binom{p+\mu-\nu}{p-s-t}_q \\
&= \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{m+p-s}_q \binom{m+p-s}{p-s}_q \\
&= \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{p-s}_q \binom{\mu-p+s}{m}_q .
\end{aligned}$$

One can see from here that for $m < 0$ we have $A = 0$ and trivially also $B = 0$.

Consider now the case $m, p \geq 0$. Then using Lemma 1.1 (a), (b) note that

$$\begin{aligned}
A &= \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{m+p-s}_q \binom{m+p-s}{p-s}_q \\
&= \sum_{s=0}^{p} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{m+p-s}_q \binom{m+p-s}{p-s}_q \\
&= \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{m+p-s}_q \binom{m+p-s}{m}_q \\
&= \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{m+p-s}_q \binom{m+p-s}{m}_q \\
&= \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu}{m}_q \binom{\mu-m}{p-s}_q \\
&= \binom{\mu}{m}_q \sum_{s\in\mathbb{Z}} q^{s(\mu-m-p+s)} \binom{m-\mu+\nu}{s}_q \binom{\mu-m}{p-s}_q = \binom{\mu}{m}_q \binom{\nu}{p}_q = B.
\end{aligned}$$

□

1.2 Partition combinatorics

We will need some tools from partition combinatorics. For details we refer to [22] and [32].

As mentioned before, we will use the term *partition* for a finite, weakly decreasing sequence of positive integers and the term *generalized partition* for a finite, weakly decreasing sequence of non-negative integers.

For a partition or generalized partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ denote by $l(\lambda)$ the *length* and with $|\lambda|$ the *weight* of the partition, that is $l(\lambda) = t$, $|\lambda| = \lambda_1 + \ldots + \lambda_t$. Let $n(\lambda) = \sum(i-1)\lambda_i$. For a partition λ with

$t = l(\lambda) \le n$ define the *generalized partition* $\lambda(n) := (\lambda_1, .., \lambda_t, 0^{n-t}) \in \mathbb{N}^n$.

For two partitions λ, μ, we shall write $\mu \subseteq \lambda$ if $\mu_i \le \lambda_i$ for all $i \ge 1$. For $\mu \subseteq \lambda$ we say that $\lambda - \mu$ is a *horizontal t-strip* if $|\lambda - \mu| := |\lambda| - |\mu| = t$ and the sequences λ and μ are interlaced, in the sense that $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \ldots$.

In $\mathbb{Z}^n$ we consider the following so-called *dominance ordering*: for $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbb{Z}^n$ let $a \le b$ if and only if $a_1 \le b_1$, $a_1 + a_2 \le b_1 + b_2$, ..., $a_1 + ... + a_{n-1} \le b_1 + ... + b_{n-1}$ and $a_1 + ... + a_n \le b_1 + ... + b_n$. In the particular case of $a_1 + ... + a_n = b_1 + ... + b_n$ we will use the notation $a \trianglelefteq b$. Note that the definitions above are naturally valid also for partitions and generalized partitions.

The operator w will arrange the components of any element in $\mathbb{Z}^n$ in decreasing order and for $1 \le i < j \le n$ and $k \in \mathbb{N}^*$ we define the operator $R^k_{i,j}(a_1, ..., a_n) = (a_1, ..., a_i - k, ..., a_j + k, ..., a_n)$. Then we have:

Lemma 1.2. *Let $a_1 \ge ... \ge a_n \ge 0$ and $k \in \mathbb{N}^*$ such that $a_i - k \ge a_j + k$. Then $wR^k_{i,j}(a_1, ..., a_n) \trianglelefteq (a_1, ..., a_n)$.*

A *tableau T* is a sequence of partitions

$$(0) = \lambda^0 \subseteq \lambda^1 \subseteq ... \subseteq \lambda^r = \lambda$$

such that $\lambda^i - \lambda^{i-1}$ is a horizontal strip for $i \in \{1, \ldots, r\}$. The partition λ is called the *shape* of the tableau T and the sequence $(|\lambda^1 - \lambda^0|, ..., |\lambda^r - \lambda^{r-1}|)$ the *weight* of T.

For two partitions λ, μ the *Kostka number* $K_{\lambda\mu}$ is the number of tableaux of shape λ and weight μ. The following lemma is well-known (see [50] Proposition 7.10.5 or [22] page 26).

Lemma 1.3. *For given partitions λ, μ with $|\lambda| = |\mu|$ the Kostka number $K_{\lambda\mu} \ne 0$ if and only if $\mu \trianglelefteq \lambda$.*

Finally we say a few words about the *Littlewood-Richardson coefficients* $c^{\nu}_{\lambda\mu}$, where λ, μ, ν are partitions. They are the structure constants for the product in the ring of symmetric functions with respect to the basis of Schur functions, that is, we have $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$. There are several other interesting combinatorial definitions for these numbers (see [22] Section 5). They also appear as leading coefficients in classical Hall polynomials (see Chapter 2 and [32] Chapter II).

1.3 Tame quivers and roots

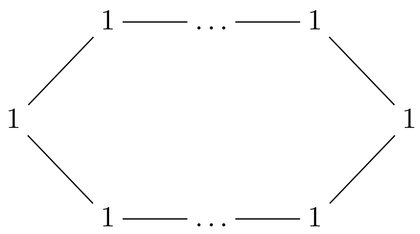
Let $Q = (Q_0, Q_1)$ be a connected, acyclic quiver of tame type with vertex set Q_0 and arrow set Q_1. Here *acyclic* means that Q has no oriented cycles or loops and *tame type* refers to the fact that the underlying undirected graph of Q is a so-called Euclidean (affine) diagram of type $\widetilde{\mathbb{A}}_m$ $(m \ge 1)$, $\widetilde{\mathbb{D}}_m$ $(m \ge 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$.

Suppose that the vertex set Q_0 has n elements and for an arrow $\alpha \in Q_1$ we denote by $t(\alpha), h(\alpha) \in Q_0$ the tail and head of α.

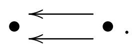
The *Euler form* of Q is a bilinear form on $\mathbb{Z}Q_0 \cong \mathbb{Z}^n$ given by $\langle x, y\rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{t(\alpha)} y_{h(\alpha)}$. Its quadratic form q_Q (called *Tits form*) is independent from the orientation of Q and exactly in the tame cases is positive semidefinite with radical $\{a \in \mathbb{Z}Q_0 | q_Q(a) = 0\} = \mathbb{Z}\delta$, where δ is the minimal radical vector of the Tits form (which is also the *minimal positive imaginary root* of the corresponding Kac-Moody root system (see [29])). The defect of $x \in \mathbb{Z}Q_0$ is then $\partial x = \langle \delta, x\rangle = -\langle x, \delta\rangle$, the *absolute defect* being the absolute value $|\partial x|$ (see Section 7 in [16], this is equivalent with the definition of the defect in [18] page 11). The vectors $a \in \mathbb{N}Q_0$ with $q_Q(a) = 1$ are called *positive real roots*. The set of positive real roots of Q is orientation independent, however the defect of these roots depend on the orientation. Here we will use the componentwise partial order on $\mathbb{Z}Q_0 \cong \mathbb{Z}^n$ defined by $x = (x_i) \leq (y_i) = y$ if $y_i - x_i \geq 0$ for $i \in Q_0$.

Below is the list of all the Euclidean diagrams marking each vertex $i \in Q_0$ with the component δ_i of the minimal radical vector δ (the number of vertices being n).

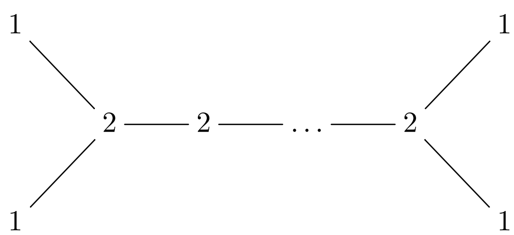
- Type $\widetilde{\mathbb{A}}_m$, with $m \geq 1$ (the case $m = 0$ having only cyclic orientation) and $n = m + 1$:



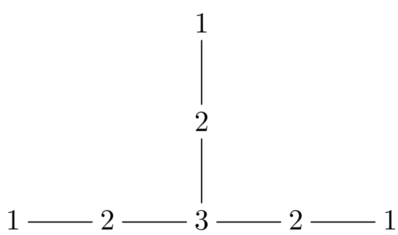
- The *Kronecker quiver* is the particular quiver of type $\widetilde{\mathbb{A}}_1$ with non-cyclic orientation:



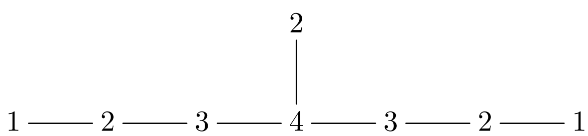
- Type $\widetilde{\mathbb{D}}_m$, with $m \geq 4$ and $n = m + 1$:



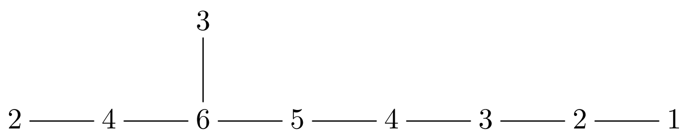
- Type $\widetilde{\mathbb{E}}_6$, with $n = 7$:



- Type $\widetilde{\mathbb{E}}_7$, with $n = 8$:



- Type $\widetilde{\mathbb{E}}_8$, with $n = 9$:



1.4 Tame hereditary algebras. Representations of tame quivers

Let k be a field and consider the path algebra kQ of the quiver Q. This means that the k-basis of kQ is formed by all the possible paths in the quiver and the multiplication is defined via concatenation of paths. The algebra kQ is a finite dimensional *tame hereditary algebra* and in fact all connected elementary tame hereditary algebras have this form. For details on the exact meaning of tameness we refer for example to [19].

The category of finite dimensional right modules (which is abelian and Krull-Schmidt) is denoted by mod-kQ, where $[M]$ is the isomorphism class, $|M|$ the length of $M \in$ mod-kQ and $uM = M \oplus \cdots \oplus M$ (u-times). For two modules $M, M' \in$ mod-kQ (in some cases) we will denote a monomorphism by $M' \hookrightarrow M$ and an epimorphism by $M \twoheadrightarrow M'$. We say that a module M' is a *subfactor* of M if (up to isomorphism) M' is the factor module of a submodule of M (that is, there exists a module L with $M' \twoheadleftarrow L \hookrightarrow M$) or equivalently (up to isomorphism) M' is the submodule of a factor module of M (that is, there is a module N such that $M \twoheadrightarrow N \hookleftarrow M'$). We will call the modules L and N *linking modules*.

For k finite and modules $X, Y \in$ mod-kQ we will denote by s_X^Y the number of submodules of Y isomorphic to X, by f_X^Y the number of submodules of Y with factor isomorphic to X, by m_X^Y the number of monomorphisms from X to Y, by e_X^Y the number of epimorphisms from Y to X, by α_X the number of automorphisms of X and by h_{XY} the number of morphisms from X to Y. Clearly $m_X^Y = \alpha_X s_X^Y$ and $e_X^Y = \alpha_X f_X^Y$.

The category mod-kQ can and will be identified with the category rep-kQ of the finite dimensional k-representations of the quiver Q. Recall that a k-representation of Q is defined as a set of finite dimensional k-spaces $\{V_i | i \in Q_0\}$ corresponding to the vertices, together with k-linear maps $V_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ corresponding to the arrows $\alpha \in Q_1$. The *dimension of a module* $M = (V_i, V_\alpha) \in$ mod-kQ = rep-kQ is then $\underline{\dim} M = (\dim_k V_i)_{i \in Q_0} \in \mathbb{Z}Q_0$. Following [18] we will call an indecomposable module in mod-kQ of *discrete dimension type* if its dimension is not of the form $t\delta$ and of *continuous dimension type* if its dimension is of the form $t\delta$.

Let $S(i)$, $P(i)$ and $I(i)$ be the indecomposable simple, projective and injective module corresponding to the vertex i and consider the *Cartan matrix* C_Q with the j-th column being $\underline{\dim} P(j)$. The *Coxeter matrix* is defined as $\Phi_Q = -C_Q^t C_Q^{-1}$. Then $\delta \Phi_Q^t = \delta$ and the Euler form satisfies $\langle a, b \rangle = a C_Q^{-t} b^t = -\langle b, a\Phi_Q^t \rangle = \langle a\Phi_Q^t, b\Phi_Q^t \rangle$, where $a, b \in \mathbb{Z}Q_0$ (see in [2], pp. 93). Moreover (because our algebra is hereditary) for two modules $X, Y \in$ mod-kQ the Euler form is:

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \dim_k \mathrm{Hom}(X, Y) - \dim_k \mathrm{Ext}^1(X, Y).$$

The indecomposable modules in mod-kQ are of three types: preprojectives (having negative defect), preinjectives (having positive defect) and regulars (having zero defect) (see [16] Lemma 2). By a theorem of Dlab and Ringel (see Theorem in [18], pp. 2) we know that the indecomposable preprojectives, preinjectives and regulars of discrete dimension type, correspond bijectively (via their dimension vector) to the positive real roots of Q. So the indecomposables which are of discrete dimension type, are uniquely determined by their dimension vector. As mentioned above, positive real roots do not depend on the orientation of Q, however the type (i.e. preprojective, preinjective or regular) of the indecomposables of discrete dimension type is orientation dependent (since the defect is orientation dependent).

Remark 1.1. (a) In case an indecomposable is of discrete dimension type having as dimension the positive real root x, we will use the notation $M(x)$ or specifically $P(x), R(x)$, respectively $I(x)$ in the cases $\partial x < 0$, $\partial x > 0$, respectively $\partial x = 0$, emphasizing the fact that the indecomposable is preprojective, preinjective, respectively regular (non-homogeneous).

(b) If we need to emphasize the base field k we will use the notations M^k or $M^k(x)$.

Consider the Auslander-Reiten translates $\tau = D\operatorname{Ext}^1(-, kQ)$ and $\tau^{-1} = \operatorname{Ext}^1(D(kQ), -)$, where $D = \operatorname{Hom}_k(-, k)$. We then have the following functorial isomorphism (called *Auslander-Reiten formula*): $\operatorname{Ext}^1(X, Y) \cong D\operatorname{Hom}(\tau^{-1}Y, X) \cong D\operatorname{Hom}(Y, \tau X)$. An indecomposable module M is *preprojective* (*preinjective*) if there exists a positive integer m such that $\tau^m(M) = 0$ ($\tau^{-m}(M) = 0$). Otherwise M is said to be *regular*. Note that an indecomposable module M is preprojective (preinjective, regular) if and only if $\partial M < 0$ ($\partial M > 0$, $\partial M = 0$). A module is called preprojective (preinjective, regular) if all its indecomposable components are preprojective (preinjective, regular). We will use the notation P, I and R in these cases. Also note that for M indecomposable non-projective, non-injective we have $\underline{\dim}\tau M = \underline{\dim} M \cdot \Phi_Q^t$ and $\underline{\dim}\tau^{-1}M = \underline{\dim} M \cdot \Phi_Q^{-t}$.

If $P = \tau^{-m}P(i)$ is a preprojective indecomposable, then

$$\partial P = -\langle \underline{\dim}\tau^{-m}P(i), \delta\rangle = -\langle \underline{\dim}P(i)(\Phi_Q^t)^{-m}, \delta(\Phi_Q^t)^{-m}\rangle = -\langle \underline{\dim}P(i), \delta\rangle$$

$$= -\underline{\dim}P(i)C_Q^{-t}\delta^t = -\underline{\dim}S(i)\delta^t = -\delta_i,$$

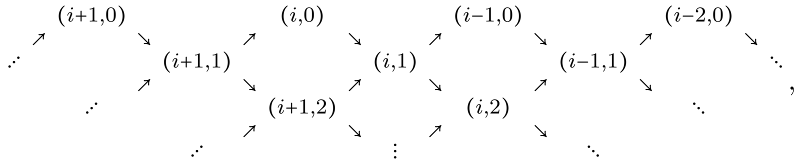
where δ_i is the i-th component of the minimal radical vector δ. This means that the possible defects of indecomposable preprojectives are -1 in the $\widetilde{\mathbb{A}}_m$ case, -1, -2 in the $\widetilde{\mathbb{D}}_m$ case, -1, -2, -3 in the $\widetilde{\mathbb{E}}_6$ case, -1, -2, -3, -4 in the $\widetilde{\mathbb{E}}_7$ case and -1, -2, -3, -4, -5, -6 in the $\widetilde{\mathbb{E}}_8$ case.

In case P is an indecomposable preprojective, then there is a unique indecomposable preprojective $P(+n\delta)$ with dimension $\underline{\dim}P(+n\delta) = \underline{\dim}P + n\delta$. Observe that $\partial P(+n\delta) = \partial P$. If $\underline{\dim}P > n\delta$ (componentwise), then there is a unique indecomposable preprojective $P(-n\delta)$ with dimension $\underline{\dim}P(-n\delta) = \underline{\dim}P - n\delta$. Note that $\partial P(-n\delta) = \partial P$. A similar statement is true for preinjectives.

Consider the $\widetilde{\mathbb{A}}_\infty$ quiver

$$\mathbb{A}_\infty : \quad 0 \to 1 \to 2 \to 3 \to \cdots$$

and $\mathbb{Z}\widetilde{\mathbb{A}}_\infty$ the associated *translation quiver*, having the form



where the translation τ is defined by $\tau(i,j) = (i+1,j)$ for $i \in \mathbb{Z}, j \in \mathbb{N}$.

For an integer $m \geq 1$, let $\mathbb{Z}\mathbb{A}_\infty/(\tau^m)$ be the orbit translation quiver obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex x of $\mathbb{Z}\mathbb{A}_\infty$ with $\tau^m x$ and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with $\tau^m x \to \tau^m y$. Then $\mathbb{Z}\mathbb{A}_\infty/(\tau^m)$ is a stable translation quiver consisting of τ-periodic vertices of period m, called a *stable tube of rank* m. The set of vertices of a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^m)$ having exactly one immediate predecessor or one immediate successor is said to be the mouth of $\mathcal{T}$.

The category of regular modules is an abelian, exact subcategory which decomposes into a direct sum of serial categories with Auslander-Reiten quiver of the form $\mathbb{Z}\widetilde{\mathbb{A}}_\infty/(\tau^m)$, thus a stable tube of rank m, where the translation is the Auslander-Reiten translation. These tubes are indexed by the (scheme theoretic) closed points of the projective line $\mathbf{P}^1(k)$ (the finite field case is discussed in [64] Sections 2.1 and 2.2 or derived equivalently in [4]). More precisely, if we put homogeneous coordinates $(x:y)$ on $\mathbf{P}^1(k)$, the two affine open subsets $U' = \{(x:y) | x \neq 0\}$ and $U'' = \{(x:y)|y \neq 0\}$ cover $\mathbf{P}^1(k)$, and the formulae $z = y/x$ and $z^{-1} = x/y$ define coordinates on U' and U'' respectively, the rings $k[z]$ and $k[z^{-1}]$ being the respective rings of regular functions on U' and U''. Then a closed point a of $\mathbf{P}^1(k)$ is the zero locus of an irreducible homogeneous polynomial $P \in k[X,Y]$. If P is proportional to the polynomial X, then the closed point is the point at infinity ∞. If P is not proportional to X, then a can be viewed as the zero locus in U' of the irreducible polynomial $P(1,z) \in k[z]$. In any case, a determines P up to a non-zero scalar, and the degree $\deg a$ of a is defined as the degree of P (see [4]).

A tube of rank 1 is called *homogeneous*, otherwise it is called *non-homogeneous*. We have at most 3 non-homogeneous tubes which are all indexed by points of degree 1. Moreover, if Q is a tree (not of type $\widetilde{\mathbb{A}}_m$) we have exactly 3 non-homogeneous tubes.

Recall that over a finite field k with q elements the number of closed points of $\mathbf{P}^1(k)$ having degree 1 is $q+1$ and there are $\phi_d(q) := N(q,d) = \frac{1}{d}\sum_{d'|d}\mu(\frac{d}{d'})q^{d'}$ closed points of degree $d \geq 2$, where μ is the Möbius function. Note that this is the number of monic irreducible polynomials of degree $d \geq 2$ over the field k with q elements.

We assume that the non-homogeneous tubes are labeled by some subset of $\mathcal{S} \subseteq \{0,1,\infty\}$. In case Q is a tree (thus $\mathcal{S} = \{0,1,\infty\}$), the labeling takes the following form:

- if Q is of type $\widetilde{\mathbb{D}}_m$ then $(\operatorname{rank}\mathcal{T}_0, \operatorname{rank}\mathcal{T}_1, \operatorname{rank}\mathcal{T}_\infty) = (2, m-2, 2)$;
- if Q is of type $\widetilde{\mathbb{E}}_6$ then $(\operatorname{rank}\mathcal{T}_0, \operatorname{rank}\mathcal{T}_1, \operatorname{rank}\mathcal{T}_\infty) = (3,3,2)$;
- if Q is of type $\widetilde{\mathbb{E}}_7$ then $(\operatorname{rank}\mathcal{T}_0, \operatorname{rank}\mathcal{T}_1, \operatorname{rank}\mathcal{T}_\infty) = (3,4,2)$;
- if Q is of type $\widetilde{\mathbb{E}}_8$ then $(\operatorname{rank}\mathcal{T}_0, \operatorname{rank}\mathcal{T}_1, \operatorname{rank}\mathcal{T}_\infty) = (3,5,2)$.

The homogeneous tubes are labeled by the closed points of the scheme $\mathbb{H}_k = \mathbb{H}_\mathbb{Z} \otimes k$ for the open integral subscheme $\mathbb{H}_\mathbb{Z} = \mathbb{P}^1_\mathbb{Z} \setminus \mathcal{S}$. Let $\mathbb{H}_k(k)$ be the set of degree 1 points of $\mathbb{H}_k$ (i.e. the set of k-points).

Note that for k finite with q elements $\phi_1(q) := |\mathbb{H}_k(k)|$ equals $q+1$, q or $q-1$ in the $\widetilde{\mathbb{A}}_m$ case and $q-2$ for other tame quivers. So if k has 2 elements and the quiver is not of $\widetilde{\mathbb{A}}_m$ type, there are no homogeneous modules of dimension δ.

The simple objects in the category of regular modules are called *regular simple modules*. As it was mentioned above, any indecomposable regular module is regular uniserial and hence it is uniquely determined by its regular socle and regular length, and also by its regular top and regular length.

In case of a homogeneous tube $\mathcal{T}_a$, we have a single regular simple denoted by $R(1,a)$ with $\underline{\dim} R(1,a) = (\deg a)\delta$, which lies on the "mouth" of the tube. Here $R(t,a)$ will denote the indecomposable regular with regular socle $R(1,a)$ and regular length t. Thus $R(1,a) \subset R(2,a) \subset \cdots \subset R(t,a)$. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ let $R(\lambda, a) = R(\lambda_1, a) \oplus \cdots \oplus R(\lambda_n, a)$. Note that the homogeneous modules of dimension δ are up to isomorphism of the form $R(1,a)$, with $a \in \mathbb{H}_k(k)$.

The dimension of a non-homogeneous regular indecomposable is either a positive root (and this determines it uniquely up to isomorphism) or of the form $t\delta$, in which case we need to know its regular-socle (or regular-top) in order to be uniquely determined. In case of a non-homogeneous tube $\mathcal{T}_e$ of rank $m > 1$ on the mouth of the tube we have m regular simples denoted by ${}^iR(1,e)$, $i = 1, \dots, m$ (where their dimension is orientation dependent). We will denote by ${}^iR(t,e)$ the indecomposable regular with regular socle ${}^iR(1,e)$ and regular length t.

Note that

$$\tau({}^iR(1,e)) = {}^{i-1}R(1,e) \text{ for } i \geq 2,\ \tau({}^1R(1,e)) = {}^mR(1,e),\ \sum_{i=1}^{m} \underline{\dim}\, {}^iR(1,e) = \delta,$$

$$\text{and } \lfloor \tfrac{t}{m} \rfloor \delta \leq \underline{\dim}\, {}^iR(t,e) < (\lfloor \tfrac{t}{m} \rfloor + 1)\delta.$$

We will call the preinjective, preprojective and non-homogeneous regular indecomposable modules of *discrete type*. The homogeneous regular indecomposable modules will be called of *continuous type*.

The following well known lemmas summarize some facts in mod-kQ.

Lemma 1.4 (Chapter IX in [49], Lemma 3 in [64]). *We have the following facts:*

(a) For P preprojective, I preinjective and R regular module we have $\mathrm{Hom}(R,P) = \mathrm{Hom}(I,P) = \mathrm{Hom}(I,R) = \mathrm{Ext}^1(P,R) = \mathrm{Ext}^1(P,I) = \mathrm{Ext}^1(R,I) = 0$. It follows that the submodules of a preprojective module are always preprojective, preinjectives can project only on preinjectives, a submodule of a regular module cannot have preinjective components and a regular cannot project on preprojectives.

(b) If $a \neq a'$ and R_a (respectively $R_{a'}$) is a regular with components from the tube $\mathcal{T}_a$ (respectively $\mathcal{T}_{a'}$), then $\mathrm{Hom}(R_a, R_{a'}) = \mathrm{Ext}^1(R_a, R_{a'}) = 0$.

(c) For $\mathcal{T}_a$ homogeneous and $R(t,a)$, $R(t',a)$ indecomposables from $\mathcal{T}_a$ we have $\dim_k \mathrm{Hom}(R(t,a), R(t',a)) = \dim_k \mathrm{Ext}^1(R(t,a), R(t',a)) = \min(t,t') \deg a$. In case k is finite with q elements, the number of automorphisms is $\alpha_{R(t,a)} = q^{t \deg a} - q^{(t-1)\deg a}$.

(d) For $\mathcal{T}_e$ non-homogeneous of rank m and ${}^iR(t,e)$ an indecomposable from $\mathcal{T}_e$ such that $lm < t \leq (l+1)m$ we have $\dim_k \mathrm{End}({}^iR(t,e)) = l+1$. In case k is finite with q elements, the number of automorphisms is $\alpha_{{}^iR(t,e)} = q^{l+1} - q^l$.

(e) *For $\mathcal{T}_e$ non-homogeneous of rank m and ${}^iR(t,e)$ an indecomposable from $\mathcal{T}_e$ such that $lm \le t < (l+1)m$, we have $\dim_k \operatorname{Ext}^1({}^iR(t,e), {}^iR(t,e)) = l$.*

(f) *For P indecomposable preprojective and I indecomposable preinjective modules we have $\operatorname{End}(P) \cong k$, $\operatorname{End}(I) \cong k$ and $\operatorname{Ext}^1(P,P) = \operatorname{Ext}^1(I,I) = 0$. Thus, in case k is finite with q elements, the number of automorphisms is $\alpha_P = \alpha_I = q-1$.*

(g) *For P indecomposable preprojective of defect -1 and $R(t,a)$ a homogeneous regular indecomposable*

$$\dim_k \operatorname{Hom}(P, R(t,a)) = \langle \underline{\dim} P, t\delta \deg a\rangle = t \deg a.$$

(h) *Let $M = c_1M_1 \oplus \cdots \oplus c_tM_t$ such that M_i are pairwise nonisomorphic indecomposable modules. Then, in case k is finite with q elements, the number of automorphisms is $\alpha_M = q^m \alpha_{c_1M_1} \ldots \alpha_{c_tM_t}$ where $m = \sum_{i\neq j} c_ic_j \dim_k \operatorname{Hom}(M_i, M_j)$.*

(i) *Let $M = cN$ with N indecomposable and $\operatorname{End}(N) = k'$ a field. Then, in case k is finite with q elements, the number of automorphisms is $\alpha_M = |\operatorname{GL}_c(k')| = \prod_{1\le i\le c}(d^c - d^{i-1})$, where $d = |k'| = q^{[k':k]}$.*

(j) *For X preprojective (preinjective) indecomposable there is a unique preprojective (preinjective) indecomposable denoted by $X(+n\delta)$ of dimension $\underline{\dim} X + n\delta$. For X preprojective (preinjective) indecomposable with $\underline{\dim} X > n\delta$ there is a unique preprojective (preinjective) indecomposable denoted by $X(-n\delta)$ of dimension $\underline{\dim} X - n\delta$.*

Lemma 1.5 (Lemma 2.1 in [61], Lemma 3 in [53]). *Let P be an indecomposable preprojective with defect $\partial P = -1$, P' a preprojective module and R a regular indecomposable. Then we have:*

(a) *Every nonzero morphism $f : P \to P'$ is a monomorphism.*

(b) *For every nonzero morphism $f : P \to R$, f is either a monomorphism or $\operatorname{Im} f$ is regular. In particular, if R is regular simple and $\operatorname{Im} f$ is regular then f is an epimorphism.*

(c) *Suppose that $\underline{\dim} P > \delta$. Then P projects to a unique regular simple $R_P(1,e)$ from the mouth of each non-homogeneous tube $\mathcal{T}_e$, moreover $\dim_k \operatorname{Hom}(P, R_P(1,e)) = \langle \underline{\dim} P, \underline{\dim} R_P(1,e)\rangle = 1$.*

(d) *Suppose that $\underline{\dim} P < \delta$. Then, depending on its dimension, P embeds in or projects to a unique regular simple $R_P(1,e)$ from each non-homogeneous tube $\mathcal{T}_e$, moreover $\dim_k \operatorname{Hom}(P, R_P(1,e)) = \langle \underline{\dim} P, \underline{\dim} R_P(1,e)\rangle = 1$.*

(e) *For a homogeneous tube $\mathcal{T}_a$, we have $\dim_k \operatorname{Hom}(P, R(1,a)) = 1$, so let $R_P(1,a) := R(1,a)$.*

Lemma 1.6 ([28]). *We have the following:*

(a) *$k^* = k \setminus \{0\}$ acts freely on $\operatorname{Ext}^1(M,N)^* = \operatorname{Ext}^1(M,N) \setminus \{0\}$.*

(b) For k finite with q elements and d a nonnegative integer

$$\sum_{\substack{(t_a)_{a \text{ closed point in } \mathbf{P}^1(k)} \\ t_a \in \mathbb{Z},\ t_a \geq 0 \\ \sum_a t_a \deg a = d}} 1 = \frac{q^{d+1} - 1}{q - 1}.$$

A module without homogeneous regular components can be described combinatorially and field independently: the indecomposable components of discrete dimension type are uniquely determined by their dimension (which is a positive real root), the non-homogeneous regular indecomposable components of continuous type are of the form ${}^{i}R(t,e)$, thus determined by the triple (i,t,e). We denote this system of positive real roots and triples of the form (i,t,e) by μ and let $M(\mu,k)$ be the corresponding (up to isomorphism) unique module in mod-kQ.

The homogeneous regular part of a module will be described by a *Segre symbol*, that is a multiset of the form $\sigma = \{(\lambda^1, d_1), \dots, (\lambda^r, d_r)\}$, where λ^i are partitions and $d_i \in \mathbb{N}^*$.

Using the definitions above, a *decomposition symbol* is a pair $\alpha = (\mu, \sigma)$. Given a decomposition symbol $\alpha = (\mu,\sigma)$ and a field k, we define the decomposition class $S(\alpha, k)$ to be the set of isomorphism classes of modules of the form $M(\mu,k) \oplus R$, where $R = R(\lambda^1, a_1) \oplus \cdots \oplus R(\lambda^r, a_r)$ for some distinct points $a_1, \dots a_r \in \mathbb{H}_k(k)$ such that $\deg a_i = d_i$.

The following lemma will be useful:

Lemma 1.7 ([26]). *Given a decomposition symbol α, there exist universal polynomials a_α and n_α such that for any finite field k with $|k| = q$, $a_\alpha(q) = \alpha_A$ for all $A \in S(\alpha,k)$, and $n_\alpha(q) = |S(\alpha,k)|$. Moreover, a_α is a monic integer polynomial and $n_\alpha(q)$ is strictly increasing in q.*

Remark 1.2. For a positive real root x, in order to simplify the notation, we will denote by x the decomposition symbol corresponding to the (up to isomorphism unique) indecomposable having dimension x. In this case $n_x(q) = 1$.

Also we denote simply by δ the symbol $(\varnothing, \{((1),1)\})$ corresponding to homogeneous modules of dimension δ. In case a decomposition symbol is of the form $(\varnothing, \sigma)$, where σ is a Segre symbol (thus we have only homogeneous regular components) then it will be identified with the Segre symbol σ.

1.5 The Kronecker case

The easiest tame case is the Kronecker case, when the quiver (denoted here by K) is of acyclic type $\widetilde{\mathbb{A}}_1$ thus of the form:

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2\ .$$

Note that here we labeled the vertices with 1 and 2 and the arrows with α and β.

In this special case $\delta = (1,1)$ and the indecomposable preprojectives are (up to isomorphism) the projectives $P_0 = S(1) = P(1)$, $P_1 = P(2)$ respectively their translates $P_{2n} = \tau^{-n} P_0$ and $P_{2n+1} = \tau^{-n} P_1$. Dually, the indecomposable preinjectives are the injectives $I_0 = S(2) = I(2)$, $I_1 = I(1)$ respectively their translates $I_{2n} = \tau^n I_0$ and $I_{2n+1} = \tau^n I_1$. Note that $\underline{\dim} P_n = (n+1, n)$, $\underline{\dim} I_n = (n, n+1)$ and as representations:

$$P_n := k^{n+1} \underset{\begin{pmatrix} 0 \\ I \end{pmatrix}}{\overset{\begin{pmatrix} I \\ 0 \end{pmatrix}}{\leftleftarrows}} k^n ,$$

$$I_n := k^n \underset{\begin{pmatrix} 0 & I \end{pmatrix}}{\overset{\begin{pmatrix} I & 0 \end{pmatrix}}{\leftleftarrows}} k^{n+1} .$$

As we have mentioned before, we don't have non-homogeneous tubes in the Kronecker case, so the homogeneous tubes are parameterized by the (scheme theoretic) closed points of the projective line $\mathbf{P}^1(k)$, the set of points of degree 1 being $\mathbb{H}_k(k) = k \cup \{\infty\}$, thus in this case $\phi_1(q) := |\mathbb{H}_k(k)| = q + 1$. Viewed as representations, the indecomposable regulars on tubes of degree 1 are the following:

$$R(t, \infty) = k[X]/(X^t) \underset{id}{\overset{X}{\leftleftarrows}} k[X]/(X^t) \text{ , } t \geq 1$$

$$R(t, a) = k[X]/((X-a)^t) \underset{X}{\overset{id}{\leftleftarrows}} k[X]/((X-a)^t) \text{ , } t \geq 1, \ a \in k.$$

Regarded as representations, the indecomposable regulars on a tube $\mathcal{T}_a$ with $\deg a > 1$ are of the form

$$R(t, a) = k[X]/(\varphi_a(X)^t) \underset{X}{\overset{id}{\leftleftarrows}} k[X]/(\varphi_a(X)^t) \text{ ,}$$

where $t \geq 1$ and $\varphi_a(X)$ is the monic irreducible polynomial of degree $\deg a > 1$ uniquely corresponding to the point a over k.

Note that for $a \in \mathbb{H}_k(k)$ we have $\underline{\dim} R(t, a) = t(\deg a)\delta = (t \deg a, t \deg a)$.

The following lemma specializes and extends Lemma 1.4 in the Kronecker case.

Lemma 1.8. *We have the following:*

(a) For $n \leq m$, we have $\dim_k \mathrm{Hom}(P_n, P_m) = m - n + 1$ and $\mathrm{Ext}^1(P_n, P_m) = 0$; otherwise $\mathrm{Hom}(P_n, P_m) = 0$ and $\dim_k \mathrm{Ext}^1(P_n, P_m) = n - m - 1$.

(b) For $n \geq m$, we have $\dim_k \mathrm{Hom}(I_n, I_m) = n - m + 1$ and $\mathrm{Ext}^1(I_n, I_m) = 0$; otherwise $\mathrm{Hom}(I_n, I_m) = 0$ and $\dim_k \mathrm{Ext}^1(I_n, I_m) = m - n - 1$.

(c) $\dim_k \mathrm{Hom}(P_n, I_m) = n + m$ and $\dim_k \mathrm{Ext}^1(I_m, P_n) = m + n + 2$.

(d) $\dim_k \mathrm{Hom}(P_n, R(t, a)) = \dim_k \mathrm{Hom}(R(t, a), I_n) = t \deg a$ and $\dim_k \mathrm{Ext}^1(R(t, a), P_n) = \dim_k \mathrm{Ext}^1(I_n, R_x(t)) = t \deg a$.

By the Krull-Remak-Schmidt theorem, every module in $M \in$ mod-kK (up to isomorphism) has the following decomposition:

$$(P_{c_1} \oplus \ldots \oplus P_{c_n}) \oplus (\oplus_{a \in \mathbb{H}_k(k)} R(\lambda^a, a)) \oplus (I_{d_1} \oplus \ldots \oplus I_{d_m}), \qquad (\star)$$

where

(1) $(c_1, ..., c_n)$ is a finite, weakly increasing sequence of nonnegative integers;

(2) λ^a is a partition for every $a \in \mathbb{H}_k(k)$ (almost all of these partitions being empty);

(3) $(d_1, ..., d_m)$ is a finite, weakly decreasing sequence of nonnegative integers.

The sequences from (1), (2), (3) above will be called *Kronecker invariants* of the module M. Using the $(\star)$ decomposition, one can see that they determine M up to isomorphism.

Let $\mathcal{R}_n$ be the set of isomorphism classes of regular modules of dimension $n\delta = (n, n)$, with indecomposable components taken from pairwise different tubes. So

$$\mathcal{R}_n = \{[R(t_1, a_1) \oplus ... \oplus R(t_r, a_r)] \,|\, r, t_1, \ldots, t_r \in \mathbb{N}^*, a_1, ..., a_r \in \mathbb{H}_k \text{ are pairwise different and } t_1 \deg a_1 + ... + t_r \deg a_r = n\}.$$

Observe that $\mathcal{R}_n = \bigcup S(\sigma, k)$, where the union is taken over all Segre symbols of the form $\sigma = \{((t_1), d_1), \ldots, ((t_r), d_r)\}$ with $r \in \mathbb{N}^*$ and $t_1 d_1 + ... + t_r d_r = n$.

We will denote by $\mathcal{R}$ the full subcategory of mod-kK having as objects regular modules with indecomposable components taken from pairwise different tubes. This means that the set of isomorphism classes of the objects in $\mathcal{R}$ is exactly $\cup_{n \in \mathbb{N}^*} \mathcal{R}_n$. Note that $\mathcal{R}$ is not extension closed.

Finally we should emphasize that the Kronecker case is particularly interesting due to a result of Beilinson which proves that the module category mod-kK of the Kronecker algebra is derived equivalent to the category $\mathrm{Coh}(\mathbf{P}^1(k))$ of coherent sheaves on the projective line. Note that the indecomposable preprojectives and preinjectives together correspond in $\mathrm{Coh}(\mathbf{P}^1(k))$ to the indecomposable locally free coherent sheaves and the indecomposable regulars correspond to the indecomposable torsion sheaves.

1.6 Sums of numbers of automorphisms associated to regular modules

We present in this section formulas for specific sums of numbers of automorphisms associated to regular modules. These formulas, interesting on their own, will play an important role in the determination of Ringel-Hall polynomials. Throughout this section k is finite with q elements and all the modules are from mod-kQ (where Q is a connected, acyclic quiver of tame type).

Proposition 1.2. *We have the following:*

(a) Let ${}^0a_n(q) = \frac{1}{q-1} \sum_{[R]} \alpha_R$, *where the sum is taken over all regular modules R of dimension $n\delta$ with indecomposable components from pairwise different homogeneous tubes. Then* ${}^0a_n(q) = q^{2n-1} - 2q^{2n-2} + 3q^{2n-3} + \cdots - (2n-2)q^2 + (2n-1)q - (n+1)$. *Also by definition* ${}^0a_0(q) = \frac{1}{q-1}$ *and* ${}^0a_{-1}(q) = 0$.

(b) *Denote by* $0 < \sigma_1 < \delta$ *the dimension of an indecomposable regular taken from the non-homogeneous tube* $\mathcal{T}$. *Let* ${}^1a_n(q) = \frac{1}{q-1}\sum_{[R]}\alpha_R$, *where the sum is taken over all regular modules* R *of dimension* $n\delta + \sigma_1$ *with indecomposable components from pairwise different homogeneous tubes and the non-homogeneous tube* $\mathcal{T}$. *Then* ${}^1a_n(q) = q^{2n} - 2q^{2n-1} + 3q^{2n-2} + \cdots - (2n)q + (n+1)$. *Note that* ${}^1a_0(q) = 1$.

(c) *Denote by* $0 < \sigma_1, \sigma_2 < \delta$ *the dimensions of two indecomposable regulars taken from two different non-homogeneous tubes* $\mathcal{T}_1$ *and* $\mathcal{T}_2$. *Let* ${}^2a_n(q) = \frac{1}{q-1}\sum_{[R]}\alpha_R$, *where the sum is taken over all regular modules* R *of dimension* $n\delta+\sigma_1+\sigma_2$ *with indecomposable components from pairwise different homogeneous tubes and a single component from each non-homogeneous tube* $\mathcal{T}_1$ *and* $\mathcal{T}_2$. *Then* ${}^2a_n(q) = q^{2n+1} - 2q^{2n} + 3q^{2n-1} + \cdots - (2n)q^2 + (2n+1)q - (n+1)$. *Note that* ${}^2a_0(q) = q-1$.

(d) *Denote by* $0 < \sigma_1, \sigma_2, \sigma_3 < \delta$ *the dimensions of three indecomposable regulars taken from the pairwise different non-homogeneous tubes* $\mathcal{T}_1$, $\mathcal{T}_2$ *and* $\mathcal{T}_3$. *Let* ${}^3a_n(q) = \frac{1}{q-1}\sum_{[R]}\alpha_R$, *where the sum is taken over all regular modules* R *of dimension* $n\delta + \sigma_1 + \sigma_2 + \sigma_3$ *with indecomposable components from pairwise different homogeneous tubes and a single component from each non-homogeneous tube* $\mathcal{T}_1$, $\mathcal{T}_2$ *and* $\mathcal{T}_3$. *Then* ${}^3a_n(q) = q^{2n+2} - 2q^{2n+1} + 3q^{2n} + \cdots - (2n+2)q + (n+1)$. *Note that* ${}^3a_0(q) = (q-1)^2$.

(e) *Consider the polynomials* $f_n = X^n - 3X^{n-1} + \cdots + (-1)^{n-1}(2n-1)X + (-1)^n(n+1)$ *for* $n \geq 1$, $f_0 = 1$, $f_{-n} = 0$. *Then*

$$\begin{aligned} {}^0a_n(q) - {}^3a_{n-2}(q) &= f_{2n-1}(q), \\ {}^3a_{n-1}(q) - {}^0a_n(q) &= f_{2n}(q), \\ {}^1a_n(q) - {}^2a_{n-1}(q) &= f_{2n}(q), \\ {}^2a_n(q) - {}^1a_n(q) &= f_{2n+1}(q), \text{ for } n \geq 1. \end{aligned}$$

Proof. (a) For a a closed point in $\mathbb{H}_k$, we know (see Lemma 1.4 (c)) that $\alpha_{R(t,a)} = q^{t\deg a} - q^{(t-1)\deg a} =: \alpha_{t,\deg a}$. So the generating function corresponding to the sum above is $\frac{1}{q-1}N(x)$, where

$$N(x) = \prod_{a\in\mathbb{H}_k}(1 + \alpha_{t,\deg a}x^{\deg a} + \alpha_{t,2\deg a}x^{2\deg a} + \dots) = \prod_{a\in\mathbb{H}_k}\frac{1-x^{\deg a}}{1-(qx)^{\deg a}} = \prod_{d\geq 1}\left(\frac{1-x^d}{1-(qx)^d}\right)^{\phi_d(q)},$$

with $\phi_1(q) = q-2$ since we have included only homogeneous tubes and $\phi_d(q) = N(q,d)$ (see Section 1.4). It follows that $\log N(x) = \sum_{d\geq 1}\phi(d)(\log(1-x^d) - \log(1-(qx)^d))$. However by the proof of Lemma 16 in [28] we have for $\psi_1(q) = q+1 = \phi_1(q)+3$ and $\psi_d(q) = \phi_d(q)$ for $d \geq 2$.

$$\sum_{d\geq 1}\psi_d(q)(\log(1-x^d)) = \log(1-qx) + \log(1-x) = \sum_{d\geq 1}\phi_d(q)(\log(1-x^d)) + 3\log(1-x),$$

thus $\sum_{d\geq 1}\phi(d)\log\left(1-x^d\right)=\log(1-qx)-2\log(1-x)$. Similarly

$$\sum_{d\geq 1}\psi_d(q)(\log\left(1-(qx)^d\right))=\log\left(1-q^2x\right)+\log(1-qx)$$
$$=\sum_{d\geq 1}\phi_d(q)(\log\left(1-(qx)^d\right))+3\log(1-qx),$$

thus $\sum_{d\geq 1}\phi(d)\log\left(1-(qx)^d\right)=\log\left(1-q^2x\right)-2\log(1-qx)$. So we have that

$$\begin{aligned}\log N(x) &= \textstyle\sum_{d\geq 1}\phi(d)(\log\left(1-x^d\right)-\log\left(1-(qx)^d\right))\\ &= \log(1-qx)-2\log(1-x)-(\log\left(1-q^2x\right)-2\log(1-qx))\\ &= 3\log(1-qx)-2\log(1-x)-\log\left(1-q^2x\right),\end{aligned}$$

which means that $\frac{1}{q-1}N(x)=\frac{(1-qx)^3}{(1-x)^2(1-q^2x)(q-1)}$, and we only have to look at the Maclaurin sequence.

(b) Note that for an indecomposable regular ${}^iR(t,e)$ taken from a non-homogeneous tube $\mathcal{T}_e$ of rank m we have that $\alpha_{{}^iR(t,e)}=q^{u+1}-q^u$, where $u<\frac{t}{m}\leq u+1$ (see Lemma 1.4 (c)). It follows that ${}^1a_n(q)=\sum_{i=0}^n{}^0a_{n-i}(q)(q^{i+1}-q^i)=(q-1)\sum_{i=0}^n q^i\,{}^0a_{n-i}(q)$. So, we need to look at the Maclaurin sequence of the generating function $\frac{1}{q-1}N(x)\frac{q-1}{1-qx}=\frac{(1-qx)^2}{(1-x)^2(1-q^2x)}$.

(c) We proceed in the same way as in (b). It follows that

$$\begin{aligned}{}^2a_n(q) &= \sum_{\substack{i=\overline{0,n}\\ j=\overline{0,n}\\ i+j\leq n}}{}^0a_{n-i-j}(q)(q^{i+1}-q^i)(q^{j+1}-q^j)=(q-1)^2\sum_{\substack{i=\overline{0,n}\\ j=\overline{0,n}\\ i+j\leq n}}{}^0a_{n-i-j}(q)q^{i+j}\\ &= (q-1)^2\sum_{l=0}^n(l+1)q^l\,{}^0a_{n-l}(q),\end{aligned}$$

where $l=i+j$. So we need to look at the Maclaurin sequence of the generating function $\frac{1}{q-1}N(x)\frac{(q-1)^2}{(1-qx)^2}=\frac{(1-qx)(q-1)}{(1-x)^2(1-q^2x)}$.

(d) We proceed in the same way as before. It follows that

$$\begin{aligned}{}^3a_n(q) &= \sum_{\substack{i=\overline{0,n}\\ j=\overline{0,n}\\ t=\overline{0,n}\\ i+j+t\leq n}}{}^0a_{n-i-j-t}(q)(q^{i+1}-q^i)(q^{j+1}-q^j)(q^{t+1}-q^t)=(q-1)^3\sum_{\substack{i=\overline{0,n}\\ j=\overline{0,n}\\ t=\overline{0,n}\\ i+j+t\leq n}}{}^0a_{n-i-j}(q)q^{i+j+t}\\ &= (q-1)^3\sum_{l=0}^n\frac{(l+1)(l+2)}{2}q^l\,{}^0a_{n-l}(q).\end{aligned}$$

So, we need to look at the Maclaurin sequence of the generating function $\frac{1}{q-1}N(x)\frac{(q-1)^3}{(1-qx)^3}=\frac{(q-1)^2}{(1-x)^2(1-q^2x)}$.

□

1.7 Ringel-Hall algebras. Counting extensions, monomorphisms and epimorphisms

In this section k is a finite field with q elements. We consider the *rational Ringel-Hall algebra* $\mathcal{H}(kQ)$ of the algebra kQ. Its $\mathbb{Q}$-basis is formed by the isomorphism classes $[M]$ from mod-kQ and the multiplication is defined by $[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2}[M]$, where the structure constants $F^M_{N_1 N_2} = |\{U \subseteq M|\, U \cong N_2,\ M/U \cong N_1\}|$ are called *Ringel-Hall numbers*.

Note that for regular modules from a homogeneous tube $\mathcal{T}_a$ and partitions λ, μ, ν, we have $F^{R(\nu,a)}_{R(\lambda,a)R(\mu,a)} = g^\nu_{\lambda\mu}(q^{\deg a})$, where $g^\nu_{\lambda\mu} \in \mathbb{Z}[X]$ is the *classical Hall polynomial* corresponding to the partitions λ, μ, ν. We know that if for the Littlewood-Richardson coefficient we have $c^\nu_{\lambda\mu} = 0$ the polynomial $g^\nu_{\lambda\mu}$ is also 0, and in case $c^\nu_{\lambda\mu} \neq 0$ the polynomial $g^\nu_{\lambda\mu}$ has degree $n(\nu) - n(\lambda) - n(\mu)$ and leading coefficient $c^\nu_{\lambda\mu}$ (see [32] Chapter II (4.3)). It is also known that $g^\lambda_{\mu\nu} = g^\lambda_{\nu\mu}$ and $g^\lambda_{\mu(t)} = 0$ unless $\lambda - \mu$ is a horizontal t-strip. Also, there is an explicit formula for $g^\lambda_{\mu(t)}$, where $\lambda - \mu$ is a horizontal t-strip (see [32] (4.3), (4.12), (4.13)).

The Ringel-Hall algebra is associative, usually non-commutative, with unit element $[0]$, moreover with a minor modification (twist) on the multiplication it can be endowed with a comultiplication, becoming a bialgebra. The *composition subalgebra* $\mathcal{C}(kQ)$ of the Ringel-Hall algebra $\mathcal{H}(kQ)$ is generated by the isomorphism classes of simple modules. We know from [63] that in the tame case, for an indecomposable module M we have $[M] \in \mathcal{C}(kQ)$ if and only if M is exceptional (i.e. it has no self-extensions, see Section 1.10).

In case of Ringel-Hall algebras associated to quivers we know due to Ringel in [41] and Green in [24] that a generic version of the composition subalgebra, as a bialgebra, (up to the twist mentioned above) is the positive part of the corresponding Drinfeld-Jimbo quantum group.

The following formulas involving Ringel-Hall numbers are fundamental. The first formula is the *Riedtmann's formula*, the second one expresses associativity of the Ringel-Hall algebra and the last one, called *Green's formula*, is used to prove the (twisted) compatibility between the described multiplication and comultiplication.

Proposition 1.3. *For modules $N_1, N_1', N_2, N_2' \in$ mod-kQ we have:*

(a) Let $M \in$ mod-kQ be an arbitrary module. If $\mathrm{Ext}^1(N_1,N_2)_M$ denotes the set of all classes in $\mathrm{Ext}^1(N_1,N_2)$ representing an exact sequence with middle term M, then $F^M_{N_1N_2} = \frac{\alpha_M|\mathrm{Ext}^1(N_1,N_2)_M|}{\alpha_{N_1}\alpha_{N_2}|\mathrm{Hom}(N_1,N_2)|}$;

(b) $\sum_{[N]} F^M_{N_1 N} F^N_{N_2 N_3} = \sum_{[N]} F^N_{N_1 N_2} F^M_{N N_3}$, *where $M \in$ mod-kQ is arbitrary;*

(c)

$$\alpha_{N_1}\alpha_{N_2}\alpha_{N_1'}\alpha_{N_2'}\sum_{[M]} F^M_{N_1N_2}F^M_{N_1'N_2'}\alpha_M^{-1} =$$
$$= \sum_{[R],[S],[S'],[T]} q^{-\langle \underline{\dim} R, \underline{\dim} T\rangle} F^{N_1}_{RS}F^{N_1'}_{RS'}F^{N_2}_{S'T}F^{N_2'}_{ST}\alpha_R\alpha_S\alpha_{S'}\alpha_T.$$

We can see from the definition and Riedtmann's formula that the Ringel-Hall number $F^M_{N_1 N_2}$ is counting (up to automorphisms and isomorphism) the number of extensions of N_1 by N_2 with middle term M, so in fact Ringel-Hall numbers are counting specific extensions.

Remark 1.3. (a) Note that a nonzero Ringel-Hall number F^Z_{YX} equals 1 in case $\mathrm{Hom}(X,Y) = 0$. Indeed, if we had two different submodules $U, V \cong X$ in Z such that $Z/U, Z/V \cong Y$ then $U \cap V$ would be a proper submodule in U and thus we would have a nonzero morphism $X \cong U \to U/U \cap V \cong U + V/V \subseteq Z/V \cong Y$.

It follows using Lemma 1.4 that for P preprojective, I preinjective and R regular module we have $[P \oplus R \oplus I] = [P][R][I]$, so for $[M] = [P \oplus R \oplus I]$ and $[M'] = [P' \oplus R' \oplus I']$ we have

$$[M][M'] = [P][R][I][P'][R'][I'].$$

(b) Also note that if M, N and L have no projective (respectively injective) indecomposable direct summands, then $F^L_{MN} = F^{\tau L}_{\tau M \tau N}$ (respectively $F^L_{MN} = F^{\tau^{-1}L}_{\tau^{-1}M\tau^{-1}N}$).

In case of Ringel-Hall algebras corresponding to Dynkin quivers and tame quivers we know due to Ringel, Hubery (see below), respectively Deng and Ruan (see [42], [26], respectively [17]), that the structure constants of the multiplication are again polynomials in the number of elements of the base field. We will call these polynomials *Ringel-Hall polynomials*. If we are looking at Hall polynomials associated to indecomposable modules, the classical polynomials are just 0 or 1, the Ringel-Hall polynomials in the Dynkin case are also known and have degree up to 5 (see the list in [42]). However, we do not have much information about the Ringel-Hall polynomials in the tame case.

We continue by stating Hubery's theorem on the existence of Ringel-Hall polynomials in tame cases with respect to the decomposition classes.

Theorem 1.1 ([26]). *Given decomposition symbols α, β and γ, there exists a rational polynomial $F^\gamma_{\alpha\beta}$ such that for any finite field k with $|k| = q$,*

$$F^\gamma_{\alpha\beta}(q) = \sum_{\substack{A \in S(\alpha,k)\\ B \in S(\beta,k)}} F^C_{AB} \qquad \text{for all } C \in S(\gamma, k),$$

moreover,

$$n_\gamma(q) F^\gamma_{\alpha\beta}(q) = n_\alpha(q) \sum_{\substack{B \in S(\beta,k)\\ C \in S(\gamma,k)}} F^C_{AB} \qquad \text{for all } A \in S(\alpha, k),$$

$$n_\gamma(q) F^\gamma_{\alpha\beta}(q) = n_\beta(q) \sum_{\substack{A \in S(\alpha,k)\\ C \in S(\gamma,k)}} F^C_{AB} \qquad \text{for all } B \in S(\beta, k).$$

Here $n_\gamma(q) = |S(\gamma, k)|$ is the rational polynomial counting the elements of the corresponding decomposition class $S(\gamma, k)$ over the finite field k with q elements.

Remark 1.4. Consider the positive real roots x, y and the Segre symbol r. The corresponding decomposition symbols will be denoted by x, y, r (see Remark 1.2). In this case the polynomials F^z_{rx} or

F^z_{yr} will denote in our article Hubery's polynomial divided by the polynomial $n_r(q)$, which is again a polynomial.

For the task of counting the number of mono- and epimorphisms, we have a useful double, dimension-inductive formula by Ringel.

Proposition 1.4 (Ringel, [40]). *Let $X, Y, Z \in$ mod-kQ. Then we have the following formulas:*

$$m^Y_X = h_{XY} - \sum_{Z,\ \underline{\dim} Z < \underline{\dim} X} f^X_Z \alpha_Z s^Y_Z,$$

$$e^Y_X = h_{YX} - \sum_{Z,\ \underline{\dim} Z < \underline{\dim} X} f^Y_Z \alpha_Z s^X_Z.$$

This can be connected with Ringel-Hall numbers, since $s^Y_X = \sum_{[Z]} F^Y_{ZX}$ and $f^Y_X = \sum_{[Z]} F^Y_{XZ}$.

1.8 The Ringel-Hall algebra over the Kronecker quiver

The structure of the Ringel-Hall algebra in the Kronecker case is well-known due to the work of Baumann-Kassel in [4], Zhang in [62] and Szántó in [52, 51].

Let k be a finite field with q elements and consider the Ringel-Hall algebra $\mathcal{H}(kK)$ of the Kronecker algebra kK and its composition subalgebra $\mathcal{C}(kK)$. We will use the notations from Section 1.5.

Define $R_n = \sum_{[X] \in \mathcal{R}_n} \alpha_X [X]$, and for a partition $\lambda = (\lambda_1, \ldots, \lambda_u)$ let $R_\lambda = R_{\lambda_1} \ldots R_{\lambda_u}$. Also note that $g^\nu_{\lambda\mu} \in \mathbb{Z}[X]$ is the classical Hall polynomial.

Theorem 1.2 ([52]). *(a) We have the formulas:*

$$[P_i][P_j] = [P_i \oplus P_j] \text{ for } i < j,$$

$$[uP_i][vP_i] = \begin{bmatrix} u+v \\ v \end{bmatrix}(q)[(u+v)P_i], \text{ where } \begin{bmatrix} u+v \\ v \end{bmatrix}(q) = \frac{(q^{u+v}-1)\ldots(q^{u+1}-1)}{(q^v-1)\ldots(q-1)},$$

$$[P_i][P_j] = q^{i-j+1}[P_i \oplus P_j] + (q^{i-j+1} - q^{i-j-1}) \sum_{u=1}^{[\frac{i-j}{2}]} [P_{i-u} \oplus P_{j+u}] \text{ for } i > j,$$

and dually

$$[I_i][I_j] = [I_i \oplus I_j] \text{ for } i > j,$$

$$[uI_i][vI_i] = \begin{bmatrix} u+v \\ v \end{bmatrix}(q)[(u+v)I_i],$$

$$[I_i][I_j] = q^{j-i+1}[I_i \oplus I_j] + (q^{j-i+1} - q^{j-i-1}) \sum_{u=1}^{[\frac{j-i}{2}]} [I_{i+u} \oplus I_{j-u}] \text{ for } i < j;$$

$$[I_{n-1-i}][P_i] = \frac{1}{q-1} R_n + q^{n-1}[P_i \oplus I_{n-1-i}];$$

$$[R(\lambda,a)][P_m] = \sum_{\substack{\mu \text{ is a partition} \\ \text{such that } \lambda-\mu \\ \text{is a horizontal strip}}} q^{\deg a|\mu|} \cdot g^{\lambda}_{\mu(|\lambda-\mu|)}(q^{\deg a}) \cdot \frac{\alpha_{R(|\lambda-\mu|,a)}\alpha_{R(\mu,a)}}{\alpha_{R(\lambda,a)}} [P_{m+|\lambda-\mu|\deg a} \oplus R(\mu,a)],$$

and dually

$$[I_m][R(\lambda,a)] = \sum_{\substack{\mu \text{ is a partition} \\ \text{such that } \lambda-\mu \\ \text{is a horizontal strip}}} q^{\deg a|\mu|} \cdot g^{\lambda}_{\mu(|\lambda-\mu|)}(q^{\deg a}) \cdot \frac{\alpha_{R(|\lambda-\mu|,a)}\alpha_{R(\mu,a)}}{\alpha_{R(\lambda,a)}} [R(\mu,a) \oplus I_{m+|\lambda-\mu|\deg a}],$$

so in particular

$$\begin{aligned}[R(t,a)][P_m] = q^{t\deg a}[P_m \oplus R(t,a)] + [P_{m+t\deg a}] + \\ + \sum_{i=1}^{t-1}(q^{(t-i)\deg a} - q^{(t-i-1)\deg a})[P_{m+i\deg a} \oplus R(t-i,a)],\end{aligned}$$

and dually

$$\begin{aligned}[I_m][R(t,a)] = q^{t\deg a}[R(t,a) \oplus I_m] + [I_{m+t\deg a}] + \\ + \sum_{i=1}^{t-1}(q^{(t-i)\deg a} - q^{(t-i-1)\deg a})[R(t-i,a) \oplus I_{m+i\deg a}],\end{aligned}$$

and

$$[tR(1,a)][P_m] = q^{t\deg a}[P_m \oplus tR(1,a)] + [P_{m+\deg a} \oplus (t-1)R(1,a)],$$

and dually

$$[I_m][tR(1,a)] = q^{t\deg a}[tR(1,a) \oplus I_m] + [(t-1)R(1,a) \oplus I_{m+\deg a}];$$

$$R_n[P_m] = q^n[P_m]R_n + (q^{2n-1} + q^{2n-2})[P_{m+n}] + \sum_{i=1}^{n-1}(q^{n+i} - q^{n+i-2})[P_{m+i}]R_{n-i},$$

and dually

$$[I_m]R_n = q^n R_n[I_m] + (q^{2n-1} + q^{2n-2})[I_{m+n}] + \sum_{i=1}^{n-1}(q^{n+i} - q^{n+i-2})R_{n-i}[I_{m+i}].$$

(b) The elements $[P_{c_1}]\dots[P_{c_s}]R_\lambda[I_{d_t}]\dots[I_{d_1}]$, where λ is a partition and $(c_s,\dots,c_1)$, $(d_t,\dots,d_1)$ are generalized partitions, form a PBW-basis in $\mathcal{C}(kK)$. The structure constants are given by formulas above.

1.9 Reflection functors

Let i be a sink in the quiver Q. Denote by σ_i the *reflection* induced by the vertex i, that is for $a \in \mathbb{Z}Q_0$, $j \neq i$ we have $(\sigma_i(a))_i = -a_i + \sum_{j_0 \in N_i} a_{j_0}$ (where N_i is the set of neighbors of i) and $(\sigma_i(a))_j = a_j$. Let $\sigma_i Q$ be the quiver obtained by reversing all arrows involving i and by Q_i the quiver having the same underlying graph as Q with all its edges pointing towards i (so i is the unique sink in Q_i). Let mod-$kQ\langle i\rangle$ be the full subcategory of modules not containing the simple module corresponding to the vertex i as a direct summand.

We consider the *reflection functors* $S_i^+ : \text{mod-}kQ \to \text{mod-}k\sigma_i Q$ and $S_i^- : \text{mod-}k\sigma_i Q \to \text{mod-}kQ$. For all details concerning reflection functors we refer the reader to [7], [18], [31] or [2] Section VII.5. The following lemma summarizes all the basic properties of reflection functors:

Lemma 1.9. *(a) For $M \in \text{mod-}kQ$ indecomposable we have $S_i^{\pm} M \neq 0$ if and only if $M \ncong S_i$. Moreover in this case $S_i^{\pm} M$ is indecomposable and $\underline{\dim} S_i^{\pm} M = \sigma_i(\underline{\dim} M)$.*

(b) The functors S_i^+, S_i^- induce quasi-inverse equivalences between $\text{mod-}kQ\langle i\rangle$ and $\text{mod-}k\sigma_i Q\langle i\rangle$.

(c) If i is a sink, $M, M' \in \text{mod-}kQ$ are indecomposable modules and $S_i^+ M' \neq 0$ then S_i^+ induces an isomorphism $\text{Ext}^1(M, M') \to \text{Ext}^1(S_i^+ M, S_i^+ M')$. Dually, if i is a source, $M, M' \in \text{mod-}kQ$ are indecomposable modules and $S_i^- M \neq 0$ then S_i^- induces an isomorphism $\text{Ext}^1(M, M') \to \text{Ext}^1(S_i^- M, S_i^- M')$.

Using the lemma above we can easily prove that:

Lemma 1.10. *For k finite and $M, N, L \in \text{mod-}kQ\langle i\rangle$, using the functor S_i^+ for a sink i and S_i^- for a source i, we have:*

(a) $\alpha_M = \alpha_{S_i^{\pm} M}$, $h_{NL} = h_{S_i^{\pm} N S_i^{\pm} L}$ and $F^L_{MN} = F^{S_i^{\pm} L}_{S_i^{\pm} M\, S_i^{\pm} N}$;

(b) $\langle \underline{\dim} N, \underline{\dim} L\rangle_Q = \langle \sigma_i(\underline{\dim} N), \sigma_i(\underline{\dim} L)\rangle_{\sigma_i Q}$, $\sigma_i(\delta) = \delta$ and $\partial S_i^{\pm} M = \partial M$. Moreover, if R is a simple homogeneous (respectively non-homogeneous) regular, then so is $S_i^{\pm} R$;

(c) In case the potential factors of an embedding $N \hookrightarrow L$ or the potential kernels of a projection $L \twoheadrightarrow M$ are in $\text{mod-}kQ\langle i\rangle$ we have

$$s^L_N = s^{S_i^{\pm} L}_{S_i^{\pm} N}, \quad f^L_M = f^{S_i^{\pm} L}_{S_i^{\pm} M}.$$

From now on suppose that Q is a tame tree (i.e. not of type $\widetilde{\mathbb{A}}_m$), i is a sink in Q and N_i is the set of neighbors of i. The following technical lemma concerning tame trees has a similar proof as Lemma 5.2 on page 279 in [2].

Lemma 1.11. *There exists a sequence $i_1, ..., i_t$ of vertices of Q different from i and not in N_i such that for each $s \in \{1, \dots, t\}$ the vertex i_s is a sink in $\sigma_{i_{s-1}} \dots \sigma_{i_1} Q$ and $\sigma_{i_t} \dots \sigma_{i_1} Q = Q_i$.*

Remark 1.5. Note that in the case of the Kronecker quiver (where vertex 1 is the sink and vertex 2 is the source) $\sigma_1 K$ is in fact K with flipped vertex numbering, so the categories $\text{mod-}k\sigma_1 K\langle 1\rangle$ and $\text{mod-}kK\langle 2\rangle$ can and will be identified. Thus in this case $T^+ = S_1^+, T^- = S_2^-$ induce exact quasi-inverse equivalences between $\text{mod-}kK\langle 1\rangle$ and $\text{mod-}kK\langle 2\rangle$. In particular if $M \in \text{mod-}kK\langle 1\rangle$ with $\underline{dim} M = (m, n)$, then $m \leq 2n$ and $\underline{dim}(T^+ M) = (n, 2n - m)$. Also if $M \in \text{mod-}kK\langle 2\rangle$ with $\underline{dim} M = (m, n)$, then $n \leq 2m$ and $\underline{dim}(T^- M) = (2m - n, m)$.

1.10 Orthogonal exceptional pairs and Schofield sequences

An indecomposable module is called *exceptional* if it has no self-extensions. This implies that the only exceptional modules in the tame hereditary case are the preprojective indecomposables, preinjective indecomposables and the non-homogeneous regular indecomposables having dimension vector less than δ (see Lemma 1.4 (c), (e), (f)). This means however (see Lemma 1.4 (d), (f)), that in the tame hereditary case the endomorphism space of an exceptional module is one dimensional. Note also that all the exceptional modules are of discrete dimension type.

A pair of indecomposable modules (Y, X) is called an *orthogonal exceptional pair* if the modules are exceptional, and are *orthogonal*, that is $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = \mathrm{Ext}^1(X, Y) = 0$. Denote by $\mathcal{F}(X, Y)$ the full subcategory of objects having filtration with factors X and Y. Observe that $\mathcal{F}(X, Y)$ is an exact, hereditary, abelian subcategory equivalent to the category of finite dimensional k-representations of the quiver having two vertices and $d = \dim_k \mathrm{Ext}^1(Y, X)$ arrows from left to right.

We know due to Hubery (see [28]) that in the tame hereditary case we have $\dim_k \mathrm{Ext}^1(Y, X) \le 2$, moreover, if equality holds then $\underline{\dim} X \oplus Y = \delta$ and $\partial Y = 1$. Thus in the case $\dim_k \mathrm{Ext}^1(Y, X) = 2$ we have $X = P$ indecomposable preprojective of defect -1 and $Y = I$ indecomposable preinjective of dimension $\delta - \underline{\dim} P$ and of defect 1. This pair (I, P) is then called a *Kronecker pair* since the category $\mathcal{F}(P, I)$ is equivalent to the category of finite dimensional k-representations of the Kronecker quiver K.

The following theorem by Schofield (for k algebraically closed) and Ringel (for k general) (see [37, 48]) makes it possible to construct exceptional modules as extensions of smaller exceptional ones which are also orthogonal. This procedure is generally called *Schofield induction*. More precisely we have:

Theorem 1.3. (Schofield, [28, 37, 48]) *If Z is exceptional but not simple, then $Z \in \mathcal{F}(X, Y)$ for some orthogonal exceptional pair (Y, X), and Z is not a simple object in $\mathcal{F}(X, Y)$. In fact, there are precisely $s(Z) - 1$ such pairs, where $s(Z)$ is the number of nonzero components in $\underline{\dim} Z$.*

Note that in the theorem above the condition requiring Z not to be a simple object in $\mathcal{F}(Y, X)$ is equivalent with the existence of an exact sequence of the form $0 \to uX \to Z \to vY \to 0$ with u, v strictly positive. Such an exact sequence will be called *Schofield sequence* corresponding to the exceptional module Z. We say that the orthogonal exceptional pair (Y, X) corresponds (is associated) to the exceptional module Z if there is a short exact sequence $0 \to X \to Z \to Y \to 0$, which is a particular Schofield sequence of Z. In this case X (respectively Y) is called *Schofield submodule (factor)* of Z.

In [60] we obtained the following characterization of Schofield sequences:

Proposition 1.5 ([60])**.** *Suppose X, Y, Z are exceptional indecomposables such that $u\underline{\dim} X + v\underline{\dim} Y = \underline{\dim} Z$. Then we have a Schofield sequence*

$$0 \longrightarrow uX \longrightarrow Z \longrightarrow vY \longrightarrow 0$$

if and only if $\langle \underline{\dim} X, \underline{\dim} Y \rangle = 0$. Moreover, in this case either $u = v = 1$ or $|u - v| = 1$ with $\partial X = -1$, $\partial Y = 1$, $\underline{\dim} X + \underline{\dim} Y = \delta$ and $\partial Z = \pm 1$. In the latter case we will speak about a special Schofield sequence. Finally u, v and thus Y are uniquely determined by X, Z. Also u, v and thus X are uniquely determined by Z, Y.

The following corollary gives a clear description of the special Schofield sequences:

Corollary 1.4 ([60]). *Let Z be an exceptional indecomposable of defect -1 (i.e. preprojective) with $\underline{\dim} Z > \delta$. Then it has a unique special Schofield sequence which is of the form*

$$0 \longrightarrow (u+1)X \longrightarrow Z \longrightarrow uY \longrightarrow 0$$

with $\partial X = -1$, $\underline{\dim} X + \underline{\dim} Y = \delta$ and thus $\underline{\dim} Z = u\delta + \underline{\dim} X$.

Dually, let Z be an exceptional indecomposable of defect 1 (i.e. preinjective) with $\underline{\dim} Z > \delta$. Then it has a unique special Schofield sequence which is of the form

$$0 \longrightarrow vX \longrightarrow Z \longrightarrow (v+1)Y \longrightarrow 0$$

with $\partial X = -1$, $\underline{\dim} X + \underline{\dim} Y = \delta$ and thus $\underline{\dim} Z = v\delta + \underline{\dim} Y$.

From now on we will use the term Schofield sequence for the non-special ones, so for those of the form $0 \to X \to Z \to Y \to 0$, with the orthogonal exceptional pair (Y, X) corresponding to Z. The corollary below follows from the proposition above:

Corollary 1.5 ([60]). *Suppose X, Y, Z are exceptional indecomposables such that $\underline{\dim} X + \underline{\dim} Y = \underline{\dim} Z$. Then we have a Schofield sequence $0 \to X \to Z \to Y \to 0$ if and only if (at least) one of the following conditions is satisfied:*

(a) $\dim_k \operatorname{Ext}^1(Y, X) = 1$;

(b) in case X and Z are not both regular, we have $\dim_k \operatorname{Hom}(X, Z) = 1$;

(c) in case X and Z are both regular they share the same regular socle;

(d) in case Y and Z are not both regular, we have $\dim_k \operatorname{Hom}(Z, Y) = 1$;

(e) in case Y and Z are both regular they share the same regular top.

The following proposition gives us the orthogonal exceptional pairs associated to exceptional modules in the $\widetilde{\mathbb{A}}_m$ case.

Proposition 1.6 ([60]). *If Q is of $\widetilde{\mathbb{A}}_m$ type then we have:*

(a) In case $\partial Z = -1$ and $\underline{\dim} Z > \delta$, all its Schofield factors are the regular exceptionals having regular top $R_Z(1, e)$ for $e \in E$. A dual statement is true for $\partial Z = 1$.

(b) The Schofield submodules of $R(t, e)$ are the regulars $R(t', e)$ with $t' < t$ and the indecomposable preprojectives P of defect -1, having dimension less than $\underline{\dim} R(t, e)$ and satisfying $R_P(1, e) = \operatorname{top} R(t, e)$ (i.e. $\dim_k \operatorname{Hom}(P, \operatorname{top} R(t, e)) = 1$).

In the other tame cases (when the quiver is a tree) we will use reflections and AR-translations in order to obtain all the orthogonal exceptional pairs over any orientation.

The following proposition is the result of the general compatibility of exact sequences with reflections.

Proposition 1.7 ([60]). *Consider the sink i, its reflection functor S_i^+ and suppose X is an indecomposable module which is not the projective simple corresponding to the sink i. If $0 \to uX \to Z \to vY \to 0$ is a Schofield sequence, then so is $0 \to uS_i^+X \to S_i^+Z \to vS_i^+Y \to 0$. The assertion remains valid also for the functor S_i^-, in the case i is a source and Y is not the injective simple corresponding to i. Moreover, we obtain the same result for the AR-translations τ (in the case X is not an indecomposable projective) and τ^{-1} (in the case Y is not an indecomposable injective).*

As we can see from the proposition above, a Schofield sequence $0 \to uX \to Z \to vY \to 0$ vanishes under the reflection S_i^+ in the case X is the projective simple corresponding to the sink i. This might suggest that the set of all the Schofield sequences associated to Z is not compatible with reflections. However for non-special Schofield sequences (orthogonal exceptional pairs) the contrary is true.

Proposition 1.8 ([60]). *Consider the sink i and its reflection functor S_i^+. If $\mathcal{M} = \{(Y_j, X_j) | j \in J\}$ is the set of all orthogonal exceptional pairs associated to Z, then $S_i^+\mathcal{M} = \{(S_i^+Y_j, S_i^+X_j) | j \in J, S_i^+X_j \neq 0\}$ is the set of all orthogonal exceptional pairs associated to S_i^+Z. Dually, if i is a source and $\mathcal{M} = \{(Y_j, X_j) | j \in J\}$ is the set of all orthogonal exceptional pairs associated to Z, then $S_i^-\mathcal{M} = \{(S_i^-Y_j, S_i^-X_j) | j \in J, S_i^-Y_j \neq 0\}$ is the set of all orthogonal exceptional pairs associated to S_i^-Z.*

In case of a canonically oriented tame tree quiver, for a given exceptional module Z all its orthogonal exceptional pairs are listed in Appendix A (taken from [60]).

We are ready now to describe the procedure of obtaining all the orthogonal exceptional pairs of any exceptional module Z over any tame quiver not of type $\widetilde{\mathbb{A}}_m$ (this type being already discussed).

Consider the tame tree quiver Q and denote by Q' the canonically oriented quiver having the same underlying graph. Then we know that there exists a sequence $i_1, \dots, i_t$ of vertices in Q such that for each $s \in \{1, \dots, t\}$ the vertex i_s is a sink in $\sigma_{i_{s-1}} \dots \sigma_{i_1} Q$ and $\sigma_{i_t} \dots \sigma_{i_1} Q = Q'$ (see [2]). We perform the following steps:

Step 1. We fix the special Schofield pair associated to Z (if $\underline{\dim} Z > \delta$ and $\partial Z = \pm 1$) using Corollary 1.4.

Step 2. For the orthogonal exceptional pairs associated to Z consider the reflected exceptional module $S_{i_t}^+ \dots S_{i_1}^+ Z$ in the canonically oriented quiver $Q' = \sigma_{i_t} \dots \sigma_{i_1} Q$.

Step 3. Using Appendix A we determine the orthogonal exceptional pairs of $S_{i_t}^+ \dots S_{i_1}^+ Z$ in Q'.

Step 4. Reflect back these orthogonal exceptional pairs using the functor $S_{i_1}^- \dots S_{i_t}^-$. By Proposition 1.8 these pairs will be all the orthogonal exceptional pairs of Z.

The following results are rather technical, however they are very useful. The first one follows directly from Proposition 7 in [60], the second one lists particular orthogonal exceptional pairs corresponding to an arbitrary preprojective indecomposable over the quiver $Q_c^{\widetilde{\mathbb{D}}_4}$ of type $\widetilde{\mathbb{D}}_4$ with all the arrows pointing towards the central vertex 5.

Lemma 1.12. *Let P be an indecomposable preprojective module of defect $-(m+n)$ and (P'', P') a corresponding orthogonal exceptional pair of indecomposable preprojective modules with $\partial P' = -m, \partial P'' = -n$. Then $(P''(\pm n\delta), P'(\pm m\delta))$ is an orthogonal exceptional pair corresponding to $P(\pm(m+n)\delta)$.*

In particular if $\partial P = -2$ and (P'', P') is a corresponding orthogonal exceptional pair of indecomposable preprojective modules with $\partial P' = \partial P'' = -1$, then $(P''(+\delta), P'(+\delta))$ is an orthogonal exceptional pair corresponding to $P(+2\delta)$.

Lemma 1.13. *Consider the quiver $Q_c^{\widetilde{\mathbb{D}}_4}$. Thus, the indecomposable preprojectives in this case are:*

- $P_2(n)$ *– the indecomposable preprojective (of defect -2) with dimension vector* $(0,0,0,0,1)+n\delta$;
- $P_1^{1j}(n)$ *(for $j=\overline{1,4}$) – the indecomposable preprojectives (of defect -1) of dimensions* $(1,0,0,0,1)+n\delta$, $(0,1,0,0,1)+n\delta$, $(0,0,1,0,1)+n\delta$, $(0,0,0,1,1)+n\delta$;
- $P_1^{2j}(n)$ *(for $j=\overline{1,4}$) – the indecomposable preprojectives (of defect -1) of dimensions* $(0,1,1,1,2)+n\delta$, $(1,0,1,1,2)+n\delta$, $(1,1,0,1,2)+n\delta$, $(1,1,1,0,2)+n\delta$.

Then, the following pairs are preprojective orthogonal exceptional pairs corresponding to $P_2(n)$:

- $(P''=P_1^{11}(\frac{n}{2}), P'=P_1^{21}(\frac{n}{2}-1))$ *in case n is even and*
- $(P''=P_1^{21}(\frac{n-1}{2}), P'=P_1^{11}(\frac{n-1}{2}))$ *in case n is odd.*

1.11 GR measures and GR submodules

The Gabriel-Roiter measure (GR measure for short) was introduced by Gabriel in order to give a combinatorial interpretation of the induction scheme used by Roiter in his proof of the first Brauer-Thrall conjecture. Ringel used it as a foundational tool for the representation theory of Artin algebras.

We list in this section some facts on GR measures and GR submodules, where GR stands for Gabriel-Roiter (see [44, 39, 38] for details).

Let $\mathcal{P}(\mathbb{N}^*)$ be the set of all subsets of $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$. A total order on $\mathcal{P}(\mathbb{N}^*)$ can be defined as follows: if I,J are two different subsets of $\mathbb{N}$, write $I<J$ if the smallest element in $(I\setminus J)\cup(J\setminus I)$ belongs to J. Using the total order above, for each $M\in$ mod-kQ let $\mu(M)$ be the maximum of the sets $\{|M_1|,|M_2|,\dots,|M_t|\}$, where $M_1\subset M_2\subset\dots\subset M_t$ is a chain of indecomposable submodules of M. Then $\mu(M)$ is *the GR measure* of M. If $M\in$ mod-kQ is indecomposable and not simple, an indecomposable submodule $U\subset M$ is called a *GR submodule* provided $\mu(M)=\mu(U)\cup\{|M|\}$, thus if and only if every proper submodule of M has GR measure at most $\mu(U)$. A monomorphism $N\to M$ between two indecomposable modules is called *GR inclusion* if $\mu(M)=\mu(N)\cup\{|M|\}$ (i.e. N is isomorphic with a GR submodule of M). It is known that the factor of a GR inclusion (called *GR factor*) is indecomposable. If $N\subset M$ is a GR inclusion, the exact sequence $0\to N\to M\to M/N\to 0$ will be called a *GR exact sequence*.

For two sets $I,J\in\mathcal{P}(\mathbb{N}^*)$ we say that J starts with I, provided $I=J$ or $I\subset J$ and for all elements $a\in I$ and $b\in J\setminus I$ we have $a<b$. Finally, for X,Y indecomposables let us denote by $\mathrm{Sing}(X,Y)$ the set of maps $X\to Y$ which are not monomorphisms.

The following proposition summarizes some known results on GR measures and GR inclusions.

Proposition 1.9. *Let $X,Y,Y_1,\dots,Y_t$ and Z be indecomposable modules.*

(a) If X is a proper submodule of Y, then $\mu(X)<\mu(Y)$.

(b) If $\mu(X)<\mu(Y)<\mu(Z)$ and there is a GR inclusion $X\to Z$, then $|Y|>|Z|$.

(c) *Suppose there is a monomorphism $f : X \to Y_1 \oplus \cdots \oplus Y_t$. If $\max\{\mu(Y_i)\}$ starts with $\mu(X)$ then there is some j such that $\pi_j f$ is injective, where $\pi_j : Y_1 \oplus \cdots \oplus Y_t \to Y_j$ is the canonical projection.*

(d) *For a GR sequence $0 \to X \xrightarrow{l} Y \xrightarrow{\pi} Z \to 0$ we have:*

(i) *X is a direct summand of all proper submodules of Y containing X; More generally, any monomorphism $f : X \to Y$ is mono-irreducible, that is for any factorization $f = f''f'$, where $f'' : X' \to Y$ is a proper monomorphism, the map $f' : X \to X'$ is a split monomorphism. In particular we can't have proper monomorphisms $X \to X'$, $X' \to Y$ with X' indecomposable.*

(ii) *There is an irreducible monomorphism $X \to M$ with M indecomposable and an epimorphism $M \to Y$ such that the composition $X \to M \to Y$ is injective;*

(iii) *Any homomorphism to Z, which is not an epimorphism, factors through π;*

(iv) *All irreducible maps to Z are epimorphisms;*

(v) *If all irreducible maps to Y are monomorphisms, then l is an irreducible map;*

(vi) *Z is a factor module of $\tau^{-1}X$ and $Z \cong \tau^{-1}X$ if and only if the GR sequence is an Auslander-Reiten sequence.*

(e) *If $X \to Y$ is a GR inclusion, then $\mathrm{Sing}(X,Y)$ is a k-subspace of $\mathrm{Hom}(X,Y)$. Moreover, if X, Y are preprojective, there is a monomorphism $X \to Y$ which is a composition of irreducible maps, so we have a path from X to Y in the preprojective component of the AR quiver.*

(f) *([45]) Suppose k is finite with q elements and let $X \hookrightarrow Y$ be a GR inclusion. Then the number s_X^Y of submodules of Y which are isomorphic to X is $\frac{q^{s-r}(q^{h-s}-1)}{(q^{e-r}-1)}$ (and $h > s \geq r, e > r$), where $e = \dim_k \mathrm{End}(X)$, $r = \dim_k \mathrm{rad}\,\mathrm{End}(X)$, $h = \dim_k \mathrm{Hom}(X,Y)$, $s = \dim_k \mathrm{Sing}(X,Y)$. In particular, for GR inclusion of preprojectives $P \hookrightarrow P'$ we have that $s_P^{P'} = \frac{q^h - q^s}{q-1}$.*

(g) *([14]) If Q is of type $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$ then any Auslander-Reiten sequence terminating at a non-injective GR factor has an indecomposable middle term.*

1.12 Connection with geometry

We begin by listing some basic facts from [16, 34] on the geometry of representations. For further details we refer to [16, 35, 34, 27].

Let k be an algebraically closed field of characteristic 0 and a finite quiver Q without oriented cycles. Consider the affine r-space $\mathbb{A}_k^r$ over k with the Zariski topology. A *locally closed subset* U in $\mathbb{A}_k^r$ is open in its closure $\overline{U}$. A non-empty locally closed subset U is *irreducible* if any non-empty subset of U which is open in U is also dense in U (i.e. its closure is U). It is well known that the affine space $\mathbb{A}_k^r$ is irreducible.

The dimension of a non-empty locally closed subset U is

$$\dim U = \sup\{n \in \mathbb{N} \mid \exists C_0 \subset C_1 \subset \ldots \subset C_n \text{ irreducible subsets closed in } U\}.$$

It is well known that $\dim U = \dim \overline{U}$, $\dim U \cup V = \max\{\dim U, \dim V\}$ and $\dim \mathbb{A}_k^r = r$.

If an algebraic group G acts on $\mathbb{A}_k^r$, then the orbits $G(x)$ are locally closed and $\overline{G(x)} \setminus G(x)$ is a union of orbits of dimension strictly smaller than $\dim G(x)$. The orbit-stabilizer theorem gives us $\dim G(x) = \dim G - \dim G_x$, where G_x is the stabilizer of the point x.

For $d \in \mathbb{N}Q_0$, consider the affine space

$$R(d) = \prod_{\alpha \in Q_1} k^{d_{h(\alpha)} \times d_{t(\alpha)}} = \mathbb{A}_k^{\sum_{\alpha \in Q_1} d_{h(\alpha)} d_{t(\alpha)}}$$

which parameterises the k-representations of Q having dimension vector d. We always identify a point $M = (M_\alpha)_{\alpha \in Q_1}$ in $R(d)$ with the corresponding representation in mod-kQ. The group

$$GL(d) = \prod_{i \in Q_0} GL_{d_i}(k)$$

acts on $R(d)$ by conjugation on the corresponding matrices, i.e.

$$(g_i)_i (M_\alpha)_\alpha = \left(g_{h(\alpha)} M_\alpha g_{t(\alpha)}^{-1} \right)_\alpha .$$

Note that $GL(d)$ is open (and non-empty) in the irreducible affine space $\mathbb{A}_k^{\sum_{i \in Q_0} d_i^2}$, thus is dense, so $\dim GL(d) = \dim \mathbb{A}_k^{\sum_{i \in Q_0} d_i^2} = \sum_{i \in Q_0} d_i^2$.

So by definition we have the following:

- The orbits $\mathcal{O}(M)$ of $GL(d)$ in $R(d)$ are in bijection with the isomorphism classes $[M]$ of k-representations of Q (modules in mod-kQ) of dimension vector d;
- The stabilizer of a point M is precisely its group of kQ-automorphisms $\operatorname{Aut}(M)$, which is open (and non-empty) in the vector space of kQ-endomorphisms $\operatorname{End}(M)$, thus is dense, so $\dim \operatorname{Aut}(M) = \dim \operatorname{End}(M)$. It follows by the orbit-stabilizer theorem that $\dim \mathcal{O}(M) = \dim GL(d) - \dim \operatorname{End}(M) = \dim_k GL(d) - \dim_k \operatorname{End}(M)$;
- $q_Q(d) = \sum_{i \in Q_0} d_i^2 - \sum_{\alpha \in Q_1} d_{h(\alpha)} d_{t(\alpha)} = \dim GL(d) - \dim R(d)$.

Using the results above, we have for a point M in $R(d)$:

Lemma 1.14. $\operatorname{codim} \mathcal{O}(M) = \dim R(d) - \dim \mathcal{O}(M) = \dim_k \operatorname{End}(M) - q_Q(d) = \dim_k \operatorname{Ext}^1(M, M)$.

This implies the following consequences:

Corollary 1.6. *We have:*

(a) For $d \neq 0$ and $q_Q(d) \leq 0$ there are infinitely many orbits in $R(d)$.

(b) $\mathcal{O}(M)$ is open if and only if M has no self-extensions.

(c) Up to isomorphism there is at most one module of dimension d without self-extensions.

Proof. (a) Observe that $\dim R(d) - \dim \mathcal{O}(M) = \dim_k \operatorname{End}(M) - q_Q(d) > 0$.

(b) Using the lemma above, we have that M has no self-extensions if and only if $\dim R(d) = \dim \mathcal{O}(M) (= \dim \overline{\mathcal{O}(M)})$. If $\mathcal{O}(M)$ is open, then since $R(d)$ is irreducible, we have that $\overline{\mathcal{O}(M)} = R(d)$, so $\dim R(d) = \dim \overline{\mathcal{O}(M)}$. Conversely, if $\dim R(d) = \dim \overline{\mathcal{O}(M)}$, then $R(d) = \overline{\mathcal{O}(M)}$, using the irreducibility of $R(d)$. But $\mathcal{O}(M)$ is locally closed, so $\mathcal{O}(M)$ is open in its closure which is $R(d)$.

(c) If $M, N \in R(d)$ are nonisomorphic without self extension, then the orbits $\mathcal{O}(M)$, $\mathcal{O}(N)$ are open and disjoint, so $\mathcal{O}(M) \subseteq R(d) \setminus \mathcal{O}(N)$, which implies $\overline{\mathcal{O}(M)} \subseteq R(d) \setminus \mathcal{O}(N)$, contradicting the irreducibility of $R(d)$.

□

Proposition 1.10 ([16]). *For a non-split exact sequence $0 \to U \to X \to V \to 0$ we have $\mathcal{O}(U \oplus V) \subseteq \overline{\mathcal{O}(X)} \setminus \mathcal{O}(X)$. So, if $\mathcal{O}(U \oplus V)$ has maximal dimension, then $\operatorname{Ext}^1(V, U) = 0$.*

For subsets $\mathcal{A} \subset R(d), \mathcal{B} \subset R(e)$, we define $\mathcal{A} * \mathcal{B} \subset R(d+e)$ to be the subset of all extensions of representations in $\mathcal{A}$ by representations in $\mathcal{B}$, that is:

$$\mathcal{A}*\mathcal{B} = \{X \in R(d+e) | \text{ there exists an exact sequence } 0 \to N \to X \to M \to 0 \text{ for some } M \in \mathcal{A}, N \in \mathcal{B}\}.$$

We know from [35] that if $\mathcal{A} \subset R(d), \mathcal{B} \subset R(e)$ are irreducible, closed subvarieties, which are stable under the corresponding group actions, then the same holds for $\mathcal{A} * \mathcal{B}$. Also if $\mathcal{C} \subset R(f)$ then we have $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$.

Using the results above, we can define the *extension monoid* $\mathcal{M}(Q)$ of Q as the set of all irreducible, closed, $GL(d)$-stable subvarieties of all $R(d)$ for $d \in \mathbb{N}Q_0$, together with the operation $*$ and the unit element $R(0)$. This monoid is naturally $\mathbb{N}Q_0$-graded by setting $\mathcal{M}(Q)_d := \{\mathcal{A} \in \mathcal{M}(Q) | \mathcal{A} \subset R(d)\}$.

Given subvarieties $\mathcal{A} \subset R(d), \mathcal{B} \subset R(e)$ from $\mathcal{M}(Q)$, we can define

$$\hom(\mathcal{B}, \mathcal{A}) := \min \{\dim_k \operatorname{Hom}(B, A) | A \in \mathcal{A}, B \in \mathcal{B}\},$$

and

$$\operatorname{ext}(\mathcal{B}, \mathcal{A}) := \hom(\mathcal{B}, \mathcal{A}) - \langle e, d \rangle.$$

Then we have the following formula:

Theorem 1.7 (Reineke). *If $\operatorname{ext}(\mathcal{A}, \mathcal{B}) = 0$ or if $\mathcal{A} = \operatorname{Rep}(\alpha)$ and $\mathcal{B} = \operatorname{Rep}(\beta)$, then*

$$\operatorname{codim} \mathcal{A} * \mathcal{B} = \operatorname{codim} \mathcal{A} + \operatorname{codim} \mathcal{B} + \operatorname{ext}(\mathcal{B}, \mathcal{A}).$$

It follows that:

Corollary 1.8 (Hubery). *Let M and N be modules such that $\dim \operatorname{Ext}^1(M, N) = 0$. Then*

$$\overline{\mathcal{O}(M)} * \overline{\mathcal{O}(N)} = \overline{\mathcal{O}(M \oplus N)}.$$

Proof. By Lemma 1.14 for $M \in R(d)$ we have $\operatorname{codim} \overline{\mathcal{O}(M)} = \dim_k \operatorname{Ext}^1(M, M)$. Using the previous theorem we obtain that $\operatorname{codim} \overline{\mathcal{O}(M \oplus N)} - \operatorname{codim} \overline{\mathcal{O}(M)} * \overline{\mathcal{O}(N)} = \dim_k \operatorname{Ext}^1(M \oplus N, M \oplus N) - \dim_k \operatorname{Ext}^1(M, M) - \dim_k \operatorname{Ext}^1(N, N) - \dim_k \operatorname{Ext}^1(N, M) = 0$. □

We say that a module N is a *degeneration* of a module M if $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$, and we denote this fact by $M \leq_{\deg} N$. Thus $\leq_{\deg}$ is a partial order on the set of isomorphism classes in mod-kQ of a given dimension. In the Dynkin and Euclidean cases one can characterize $\leq_{\deg}$ in terms of representation theory (see the work of Bongartz in [9, 8] and Zwara in [65]).

Consider the following partial orders $\leq_{\text{ext}}$ and $\leq$ on the isomorphism classes in mod-kQ. They are defined in terms of representation theory as follows:

- $M \leq_{\text{ext}} N \Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in mod-kQ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.

- $M \leq N \Leftrightarrow \dim_k \operatorname{Hom}(X, M) \leq \dim_k \operatorname{Hom}(X, N)$ holds for all modules X.

Then we have the following:

Theorem 1.9 ([9], [65])**.** *The partial orders $\leq$, $\leq_{\deg}$ and $\leq_{\text{ext}}$ are equivalent for all representations of Dynkin and Euclidean quivers.*

The following lemma taken from [10] (see also [47]) is crucial in connecting our combinatorial results obtained over finite fields with the classical Euler-Poincaré characteristic over $\mathbb{C}$.

Lemma 1.15. ([10]) *Let X be a variety defined over some ring of algebraic integers. We denote by $X(\mathbb{C})$ (respectively $X(\mathbb{F}_q)$) the set of $\mathbb{C}$-points (resp. $\mathbb{F}_q$-points) of X. Suppose that there exists a polynomial f with integral coefficients such that $|X(\mathbb{F}_q)| = f(q)$ for infinitely many prime powers q. Then the Euler-Poincaré characteristic (with compact support) of $X(\mathbb{C})$ is given by $\chi(X(\mathbb{C})) = f(1)$.*

We end this section by remarking that in the Kronecker case, taking $k = \mathbb{C}$, we have $\mathbb{H}_{\mathbb{C}}(\mathbb{C}) = \mathbf{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ and the regular indecomposables (up to isomorphism) are

$$R(t,a): \ \mathbb{C}^t \underset{I_t}{\overset{aI_t+J_t}{\longleftarrow\!\!\longleftarrow}} \mathbb{C}^t \ \text{ for } a \in \mathbb{C} \text{ and } R(t,\infty): \ \mathbb{C}^t \underset{J_t}{\overset{I_t}{\longleftarrow\!\!\longleftarrow}} \mathbb{C}^t \ ,$$

where J_t denotes a Jordan block of dimension t with eigenvalue 0 and I_t the identity matrix.

Chapter 2

Tame Ringel-Hall polynomials

We presented in the introduction the important role played by Ringel-Hall algebras in the theory of quantum groups and cluster algebras. Hence it is important to understand the structure of these algebras, and as a crucial step towards this direction, to describe explicitly the Ringel-Hall products and obtain the Ringel-Hall polynomials. As we will see, the task is not easy and it gets more and more difficult if it involves indecomposables with higher absolute defects.

Throughout the chapter we will work in the category mod-kQ, where Q is a connected, acyclic quiver of tame type and k is a finite field with q elements.

The first section lists all the Ringel-Hall polynomials associated to indecomposable modules of absolute defect up to 1. We will also describe all the Ringel-Hall products involving these indecomposables. The results of this section were published mainly in [54] and partially in [53].

The second section presents all the Hall polynomials of the form $F^{\delta}_{\delta-aa}$, where a is a positive real root of arbitrary negative defect. These polynomials correspond to the Ringel-Hall numbers of the form $F^{R(1,a)}_{IP}$, where $R(1,a)$ is a homogeneous regular simple of dimension δ and will have an application presented in the next chapter. These results appeared in [57].

The third section presents the Ringel-Hall polynomials associated to indecomposable modules of absolute defect up to 2, with one of the indecomposables having absolute defect 2. The main ingredient of the proof is a Schofield induction applied to Green's formula.

2.1 Ringel-Hall polynomials involving indecomposable modules of absolute defect up to 1

2.1.1 Reductions and main tools

Our aim is to determine the tame Ringel-Hall polynomials F^z_{xy} with $z = x + y$ and x, y, z corresponding to indecomposable modules of defect up to 1. Note that this covers all the Ringel-Hall polynomials associated to indecomposable modules in the acyclic $\widetilde{\mathbb{A}}_m$ case.

We will use the following reductions:

(a) We can choose any acyclic orientation for every tame quiver, since Ringel-Hall polynomials are reflection independent, up to a simple projective or injective module (see Lemma 1.10).

(b) Using duality arguments we can interchange preinjectives with preprojectives.

By the reductions above and Lemma 1.4 one can see that we may get nonzero F^z_{xy} Ringel-Hall polynomials in the following cases (here all the symbols x, y, z correspond to indecomposables and also $z = x + y$):

(1) $F^{r_3}_{r_2 r_1}$, where r_1, r_2, r_3 are symbols corresponding to regular indecomposables;

(2) $F^{p'}_{rp}$, where p, p' are positive real roots with $\partial p = \partial p' = -1$ (thus corresponding to preprojective indecomposables of defect -1) and r is the symbol of a regular indecomposable; or dually $F^{i'}_{ir}$, where $\partial i = \partial i' = 1$ (thus corresponding to preinjective indecomposables of defect 1) and r is the symbol of a regular indecomposable;

(3) $F^r_{i_0 p}$, where p, i_0 are positive real roots with $\partial p = -1$, $\partial i_0 = 1$ and r is the symbol of a regular indecomposable.

2.1.2 The polynomials $F^{r_3}_{r_2 r_1}$

The following proposition follows trivially using Lemma 1.4. Note that this is the case of classical Hall polynomials (so they are just 0 or 1).

Proposition 2.1. *For three symbols r_1, r_2, r_3 corresponding to regular indecomposables R_1, R_2, R_3, the Ringel-Hall polynomial $F^{r_3}_{r_2 r_1}$ is either 1 (due to uniseriality) if the regular indecomposables are from the same tube and we have an exact sequence $0 \to R_1 \to R_3 \to R_2 \to 0$, or otherwise 0.*

2.1.3 The polynomials $F^{p'}_{rp}$

We begin with a proposition which tells us more than the required polynomials.

Let $P \ncong P'$ be indecomposable preprojectives with defect -1. Then for a module X we consider the following *condition list (1)*:

(i) X is a regular module with $\underline{\dim} X = \underline{\dim} P' - \underline{\dim} P$;

(ii) if X has an indecomposable component from a tube $\mathcal{T}_a$, then the regular top of this component is the regular simple $R_{P'}(1, a)$;

(iii) the indecomposable components of X are taken from pairwise different tubes.

Using the list above the following proposition describes the Ringel-Hall numbers of the form $F^{P'}_{XP}$:

Proposition 2.2. *We have the following:*

(a) If $\mathrm{Hom}(P, P')(= \langle \underline{\dim} P, \underline{\dim} P' \rangle) = 0$ then $F^{P'}_{XP} = 0$ for every module X.

(b) If $\mathrm{Hom}(P, P')(= \langle \underline{\dim} P, \underline{\dim} P' \rangle) \neq 0$ then $F^{P'}_{XP} = 1$ for X satisfying condition list (1) and $F^{P'}_{XP} = 0$ otherwise.

Moreover, in this case $\underline{\dim} P' - \underline{\dim} P = d\delta + \sum_{e \in \mathcal{S}} \sigma_e$, where $0 \leq \sigma_e < \delta$ if nonzero is the dimension of a unique regular non-homogeneous indecomposable from $\mathcal{T}_e$ (thus a root) with top $R_{P'}(1, e)$. Also, $\dim_k \mathrm{Hom}(P, P') = \langle \underline{\dim} P, \underline{\dim} P' \rangle = d + 1$, which means that d is unique.

In case Q is a tree, thus $\mathcal{S} = \{0, 1, \infty\}$, we will use the notation $(\underline{\dim}P' - \underline{\dim}P)_{nh} := (\sigma_0, \sigma_1, \sigma_\infty)$ *and* $\mathrm{supp}(\underline{\dim}P' - \underline{\dim}P)_{nh}$ *for the number of its nonzero components.*

Proof. (a) Is trivial.

(b) *First step.* We will prove that $F^{P'}_{XP} \neq 0$ if and only if X satisfies the given condition list (1). Suppose $F^{P'}_{XP} \neq 0$. We will check the conditions (i), (ii), (iii) from the list.

Condition (i). Since $F^{P'}_{XP} \neq 0$ we have a short exact sequence $0 \to P \to P' \to X \to 0$. Then $\underline{\dim}X = \underline{\dim}P' - \underline{\dim}P$ and $\partial P' = \partial P + \partial X$, but $\partial P' = \partial P = -1$, so $\partial X = 0$. Note that X can't have preprojective components, since if P'' was such a component then $P' \twoheadrightarrow P'' \ncong P'$ which is impossible due to Lemma 1.5 (a). So X is regular.

Condition (ii). Let R be an indecomposable component of X taken from the tube $\mathcal{T}_a$. Denote by $\mathrm{top}R$ its regular top, which must be regular simple due to uniseriality. Then $P' \twoheadrightarrow X \twoheadrightarrow R \twoheadrightarrow \mathrm{top}R$, so using Lemma 1.5 we have $\mathrm{top}R \cong R_{P'}(1, a)$.

Condition (iii). Suppose $X = X' \oplus R_1 \oplus \dots \oplus R_l$, where $R_1, \dots, R_l$ are taken from the same tube $\mathcal{T}_a$. Then by Condition (ii) they have the same regular top $R_{P'}(1, a)$ and we have the monomorphism

$$0 \to \mathrm{Hom}(X, R_{P'}(1, a)) \to \mathrm{Hom}(P', R_{P'}(1, a)).$$

It follows that

$$\dim_k \mathrm{Hom}(X, R_{P'}(1, a)) \leq \dim_k \mathrm{Hom}(P', R_{P'}(1, a)) = \deg a.$$

So

$$\dim_k \mathrm{Hom}(X, R_{P'}(1, a)) = \dim_k \mathrm{Hom}(X', R_{P'}(1, a)) + \sum_{i=1}^{l} \dim_k \mathrm{Hom}(R_i, R_{P'}(1, a)) \leq \deg a$$

and $\dim_k \mathrm{Hom}(R_i, R_{P'}(1, a)) = \deg a$ for $\mathcal{T}_a$ homogeneous and $\dim_k \mathrm{Hom}(R_i, R_{P'}(1, a)) \geq 1 = \deg a$ for $\mathcal{T}_a$ non-homogeneous. It follows that $l = 1$.

Conversely, suppose now that R is an indecomposable regular module with $\underline{\dim}R < \underline{\dim}P'$ satisfying Condition (ii). By Lemma 1.5 (b) it follows that for a nonzero morphism $f : P' \to R$ the image $\mathrm{Im}\, f$ is regular. We will show that P' projects on R. Observe that if $R = R_{P'}(1, a)$, the assertion is true due to Lemma 1.5. Suppose now that R is not a regular simple.

If R is from a homogeneous tube $\mathcal{T}_a$ then $R = R(t, a)$, $\underline{\dim}R = t\delta \deg a$ and $\mathrm{Hom}(P', R) \neq 0$ since $\dim_k \mathrm{Hom}(P', R) = \langle \underline{\dim}P', t \deg a \delta \rangle = -t \deg a \partial P' = t \deg a$. Note that in case there are no epimorphisms in $\mathrm{Hom}(P', R)$, then using Lemma 1.5 (b) and the uniseriality of regulars we would have $\mathrm{Hom}(P', R) = \mathrm{Hom}(P', R(t, a)) \cong \mathrm{Hom}(P', R(t-1, a))$, a contradiction. So we have an epimorphism $P' \to R$.

If R is from a non-homogeneous tube $\mathcal{T}_e$ of rank m, then $\deg e = 1$, $R = R(t, e)$ and $\mathrm{top}R = R_{P'}(1, a) = {}^{i}R(1, e)$ (Condition (ii)). We have that $\underline{\dim}R = \underline{\dim}\,{}^{i}R(t-1, e) + \underline{\dim}(\mathrm{top}R)$, so $\dim_k \mathrm{Hom}(P', R) = \langle \underline{\dim}P', \underline{\dim}\,{}^{i}R(t-1, e)\rangle + \langle \underline{\dim}P', \underline{\dim}(\mathrm{top}R)\rangle = \dim_k \mathrm{Hom}(P', {}^{i}R(t-1, e)) + 1 > 0$. If there is no epimorphism $P' \to R$, then using uniseriality and Lemma 1.5 (b) for nonzero

$f \in \mathrm{Hom}(P', R)$ we have that $\mathrm{Im}\, f = {}^iR(l, e)$ with $1 \leq l < t$ and P' projects on $\mathrm{top}\,\mathrm{Im}\, f$ so $\mathrm{top}{}^iR(l, e) = \mathrm{top}{}^iR(t, e) = \mathrm{top}R$ (see Lemma 1.5). But this means that $t - l = sm$ with $s \geq 1$ so if $t \leq m$ we have a contradiction and if $t > m$ (as in the homogeneous case) we would have that $\mathrm{Hom}(P', {}^iR(t, e)) \cong \mathrm{Hom}(P', {}^iR(t - m, e))$ that is $0 = \langle \underline{\dim} P', \underline{\dim}{}^iR(t, e) - \underline{\dim}{}^iR(t - m, e)\rangle = \langle \underline{\dim} P', \delta\rangle = 1$, again a contradiction.

Suppose now that the module $X = R_1 \oplus \ldots \oplus R_l$ satisfies condition list (1). From the discussion above we have the epimorphisms $f_i : P' \to R_i$. Let $f : P' \to X$, $f(x) = \sum f_i(x)$ be the diagonal map. Due to Lemma 1.5 (b) we have that $\mathrm{Im}\, f$ is regular, so due to uniseriality $\mathrm{Im}\, f = R'_1 \oplus \ldots \oplus R'_l$ with $R'_i \subseteq R_i$. Since $f_i = p_i f$ are epimorphisms, we have that $R'_i = R_i$, so f is an epimorphism. Note that $\mathrm{Ker}\, f \subseteq P'$, so it is preprojective, $\partial\, \mathrm{Ker}\, f = \partial P' - \partial X = -1$, so $\mathrm{Ker}\, f$ is an indecomposable preprojective with $\underline{\dim}\, \mathrm{Ker}\, f = \underline{\dim} P$. It follows that $\mathrm{Ker}\, f \cong P$, so we have an exact sequence $0 \to P \to P' \to X \to 0$ which implies that $F^{P'}_{XP} \neq 0$.

Second step. In the case $\mathrm{Hom}(P, P') \neq 0$, we prove that $\underline{\dim} P' - \underline{\dim} P_0 = d\delta + \sum_{e \in \mathcal{S}} \sigma_e$, where $0 \leq \sigma_e < \delta$ if nonzero is the dimension of a unique regular non-homogeneous indecomposable from $\mathcal{T}_e$ (thus a root) with top $R_{P'}(1, e)$ and $\dim_k \mathrm{Hom}(P, P') = d + 1$.

From the first step we know that we have a monomorphism $P \to P'$ with factor X satisfying condition list (1). It follows that $\underline{\dim} P' - \underline{\dim} P = \underline{\dim} X = d'\delta + \sum_{e \in \mathcal{S}} \sigma'_e$, where $0 \leq \sigma_e$ if nonzero is the dimension of a regular R_e from the non-homogeneous tube $\mathcal{T}_e$ with top $R_{P'}(1, e)$. Suppose $\sigma'_e = d_e\delta + \sigma_e$, where $0 \leq \sigma_e < \delta$ and $0 \leq d_e$. If $t_e \neq 0$ then there is a unique regular R_{d_e} of dimension $d_e\delta$ from the non-homogeneous tube $\mathcal{T}_e$ which embeds into R_e; the factor will be of dimension σ_e with top $R_{P'}(1, e)$ (if $\sigma_e \neq 0$). Let $d = d' + \sum_{e \in \mathcal{S}} d_e$.

We show that $\dim_k \mathrm{Hom}(P, P') = d + 1$. Suppose first that we don't have non-homogeneous tubes, so we are in the Kronecker case. In this case $\delta = (1, 1)$, $\underline{\dim} P' - \underline{\dim} P = d\delta$ and then $\dim_k \mathrm{Hom}(P, P') = d + 1$ (see Lemma 1.8). Consider now the case when we do have non-homogeneous tubes, and suppose $d\delta + \sigma_e \neq 0$ for some $e \in \mathcal{S}$. Then there are unique regular indecomposables $R_e \in \mathcal{T}_e$ of dimension $d\delta + \sigma_e$ and top $R_{P'}(1, e)$ and $R_{e'} \in \mathcal{T}_{e'}$ of dimension $\sigma_{e'}$ and top $R_{P'}(1, e')$ for $e' \in \mathcal{S}'^* = \{e' \in \mathcal{S} | e' \neq e, \sigma_{e'} \neq 0\}$. Suppose that $\mathcal{S}^* = \{e' \in \mathcal{S} | \sigma_{e'} \neq 0\}$ and $|\mathcal{S}^*| = l$ (where we can have $l = 0$). Let $R = R_e \oplus \bigoplus_{e' \in \mathcal{S}'^*} R_{e'}$. It follows from the previous step that $F^{P'}_{RP} \neq 0$, so we have a short exact sequence $0 \to P \to P' \to R \to 0$ which induces the exact sequences

$$0 \to \mathrm{End}(P) \to \mathrm{Hom}(P, P') \to \mathrm{Hom}(P, R) \to \mathrm{Ext}^1(P, P)$$

and

$$0 \to \mathrm{End}(R) \to \mathrm{Hom}(P', R) \to \mathrm{Hom}(P, R) \to \mathrm{Ext}^1(R, R) \to \mathrm{Ext}^1(P', R).$$

We deduce using Lemma 1.4 and Lemma 1.5 that

$$\dim_k \mathrm{Hom}(P, P') = \dim_k \mathrm{Hom}(P, R) + 1 = \dim_k \mathrm{Hom}(P', R) + \dim_k \mathrm{Ext}^1(R, R) - \dim_k \mathrm{End}(R) + 1,$$

where

$$\dim_k \mathrm{Hom}(P', R) = \langle \underline{\dim} P', \underline{\dim} R\rangle = d + l,$$

$$\dim_k \operatorname{Ext}^1(R,R) = \dim_k \operatorname{Ext}^1(R_e,R_e) + \sum_{e'\in\mathcal{S}'^*} \dim_k \operatorname{Ext}^1(R_{e'},R_{e'}) = d,$$

$$\dim_k \operatorname{End}(R) = \dim_k \operatorname{End}(R_e) + \sum_{e'\in\mathcal{S}'^*} \dim_k \operatorname{End}(R_{e'}) = d + l,$$

so it results that $\dim_k \operatorname{Hom}(P,P') = d+1$.

Third step. Since by Lemma 1.5 every nonzero morphism in $\operatorname{Hom}(P,P') \neq 0$ is a monomorphism and $\alpha_P = q-1$ (using the second step) we have that the number of submodules of P' which are isomorphic to P is

$$s_P^{P'} = \frac{|\operatorname{Hom}(P,P')|-1}{\alpha_P} = \frac{q^{d+1}-1}{q-1}.$$

By the first step we have that

$$s_P^{P'} = \sum_{[X]} F_{XP}^{P'} = \sum_{\substack{[X] \\ X \text{ satisfying condition list (1)}}} F_{XP}^{P'},$$

the terms in the last sum being nonzero. We will count now the number of nonisomorphic regulars satisfying condition list (1). For $\mathcal{T}_a$ a homogeneous tube and $t \geq 1$, denote by $R_a(t)$ the regular indecomposable $R(t,a)$ and let $R_a(0) = 0$. For $\mathcal{T}_e$ ($e \in \mathcal{S}$) a non-homogeneous tube and $t \neq 0$ denote by $R_e(t)$ the unique indecomposable from $\mathcal{T}_e$ of dimension $t\delta + \sigma_e$ with top $R_{P'}(1,e)$. For $t = 0$ and $\sigma^e \neq 0$ let $R_e(0)$ be the unique indecomposable from $\mathcal{T}_e$ of dimension σ_e with top $R_{P'}(1,e)$. For $t = 0$ and $\sigma_e = 0$ let $R_e(0) = 0$. Then

$$\bigoplus_{\substack{(t_a)_{a \text{ closed point in } \mathbf{P}^1(k)} \\ t_a\in\mathbb{Z}, t_a\geq 0 \\ \sum_a t_a \deg a = d}} R_a(t_a)$$

are nonisomorphic regulars satisfying condition list (1), so by the Lemma 1.6 we have exactly $\frac{q^{d+1}-1}{q-1}$ of them. Since

$$s_P^{P'} = \frac{q^{d+1}-1}{q-1} = \sum_{\substack{[X] \\ X \text{ satisfying condition list (1)}}} F_{XP}^{P'}$$

and the number of nonzero terms in the right hand sum is $\frac{q^{d+1}-1}{q-1}$, we obtain the assertion of the proposition.

□

Specializing the proposition above to indecomposables we obtain:

Corollary 2.1. *For positive real roots $p < p'$ with $\partial p = \partial p' = -1$ (thus corresponding to indecomposable preprojectives) we have that $F_{rp}^{p'} = 1$ for any symbol r corresponding to a regular indecomposable of dimension $p' - p$ taken from a tube $\mathcal{T}_a$ and having as regular top $R_{P'}(1,a)$ (where P' is the preprojective indecomposable corresponding to the root p'). Moreover, such a symbol r exists if and only if $\langle p, p' \rangle > 0$. For any other symbol α we have $F_{\alpha p}^{p'} = 0$. This dualizes for preinjective roots with defect 1.*

2.1.4 The polynomials F_{ip}^r

This case can in fact be obtained from the previous one using a specific reflection functor and Ringel's formulas counting mono- and epimorphisms. However, we will follow a lengthier path similar to the one in the previous section.

We begin again with a proposition which tells us more than the required polynomials.

Let P be indecomposable preprojective with $\partial P = -1$ and I an indecomposable preprojective with $\partial I = 1$. Then for a module X we consider the following *condition list (2)*:

(i) X is a regular module with $\underline{\dim} X = \underline{\dim} I + \underline{\dim} P$;

(ii) if X has an indecomposable component from a tube $\mathcal{T}_a$, then the regular top of this component is the regular simple $R_P(1,a)$;

(iii) the indecomposable components of X are taken from pairwise different tubes.

Using the list above the following proposition describes the Ringel-Hall numbers of the form F_{IP}^X:

Proposition 2.3. *We have the following:*

(a) $F_{IP}^{P\oplus I} = q^{\dim_k \mathrm{Hom}(P,I)} = q^{\langle \underline{\dim} P, \underline{\dim} I\rangle}$.

(b) If $\mathrm{Ext}^1(I,P)(= \langle \underline{\dim} I, \underline{\dim} P\rangle) = 0$ *then* $F_{IP}^X = 0$ *for* $X \ncong P \oplus I$.

(c) If $\mathrm{Ext}^1(I,P)(= \langle \underline{\dim} I, \underline{\dim} P\rangle) \neq 0$ *then for* $X \ncong P\oplus I$ *we have* $F_{IP}^X = \frac{1}{q-1}\alpha_X$ *for* X *satisfying condition list (2) and* $F_{IP}^X = 0$ *otherwise.*

Moreover, in this case $\underline{\dim} I + \underline{\dim} P = d\delta + \sum_{e\in\mathcal{S}} \sigma_e$*, where* $0 \le \sigma_e < \delta$ *if nonzero is the dimension of a unique regular non-homogeneous indecomposable from* $\mathcal{T}_e$ *(thus a root) with top* $R_P(1,e)$*. Also,* $\dim_k \mathrm{Ext}^1(I,P) = \langle \underline{\dim} I, \underline{\dim} P\rangle = d+1$*, which means that* d *is unique.*

Proof. (a) It follows immediately from Riedtmann's formula. Indeed, trivially $|\mathrm{Ext}^1(I,P)_{P\oplus I}| = 1$ and by Lemma 1.4 we get $\mathrm{Hom}(I,P) = 0$, thus

$$F_{IP}^{P\oplus I} = \frac{\alpha_{P\oplus I}|\mathrm{Ext}^1(I,P)_{P\oplus I}|}{\alpha_I\alpha_P|\mathrm{Hom}(I,P)|} = \frac{\alpha_{P\oplus I}}{\alpha_I\alpha_P} = q^{\dim_k \mathrm{Hom}(P,I)}.$$

(b) Is trivial.

(c) *First step.* Let $\mathrm{Ext}^1(I,P) \neq 0$ and $X \ncong P\oplus I$. We will prove that $F_{IP}^X \neq 0$ if and only if X satisfies the given condition list (2). Suppose $F_{IP}^X \neq 0$. We will check the conditions (i), (ii), (iii) from the list.

Condition (i). Since $F_{IP}^X \neq 0$, we have a short exact sequence $0 \to P \xrightarrow{f} X \xrightarrow{g} I \to 0$. Then $\underline{\dim} X = \underline{\dim} I + \underline{\dim} P$ and $\partial X = \partial P + \partial I = 0$. Suppose $X = P' \oplus R \oplus I'$ (where P', R and I' are preprojective, preinjective and regular modules). Note that $p_{P'}f : P \to P'$ must be nonzero, so it is a monomorphism (see Lemma 1.5) which means that $\underline{\dim} P \le \underline{\dim} P'$. Dually, $fq_{I'} : I' \to I$ must be nonzero, so it is an epimorphism, which means that $\underline{\dim} I \le \underline{\dim} I'$. But $\underline{\dim} P + \underline{\dim} I = \underline{\dim} P' + \underline{\dim} R + \underline{\dim} I'$ which implies that $R = 0$ and $p_{P'}f$, $fq_{I'}$ are isomorphisms, so $X \cong P\oplus I$ is a contradiction. This means that X is regular.

Condition (ii). Let R_a be an indecomposable component of X taken from the tube $\mathcal{T}_a$. Denote by $\text{top}R_a$ its regular top which is regular simple. Then from the short exact sequence $0 \to P \to X \to I \to 0$ we obtain the exact sequence

$$0 \to \text{Hom}(I, \text{top}R_a) \to \text{Hom}(X, \text{top}R_a) \to \text{Hom}(P, \text{top}R_a).$$

Since $\text{Hom}(I, \text{top}R_a) = 0$, if $\text{Hom}(P, \text{top}R_a) = 0$, then $\text{Hom}(X, \text{top}R_a) = 0$, which is a contradiction. So $\text{Hom}(P, \text{top}R_a) \neq 0$ meaning that $\text{top}R_a = R_P(1, a)$.

Condition (iii). Suppose $X = X' \oplus R_1 \oplus \dots \oplus R_l$ where $R_1, \dots, R_l$ are indecomposables taken from the same tube $\mathcal{T}_a$. Then by Condition (ii) they have the same regular top $R_P(1, a)$ and we have the monomorphism $0 \to \text{Hom}(X, R_P(1, a)) \to \text{Hom}(P, R_P(1, a))$. This because $\text{Hom}(I, R_P(1, a)) = 0$. It follows that $\dim_k \text{Hom}(X, R_P(1, a)) \leq \dim_k \text{Hom}(P, R_P(1, a)) = \deg a$, so

$$\dim_k \text{Hom}(X, R_P(1, a)) = \dim_k \text{Hom}(X', R_P(1, a)) + \sum_{i=1}^{l} \dim_k \text{Hom}(R_i, R_P(1, a)) \leq \deg a.$$

But $\dim_k \text{Hom}(R_i, R_P(1, a)) = \deg a$ for $\mathcal{T}_a$ homogeneous and $\dim_k \text{Hom}(R_i, R_P(1, a)) \geq 1 = \deg a$ for $\mathcal{T}_a$ non-homogeneous, so we get that $l = 1$.

Conversely, suppose that X is a satisfying condition list (2), so let $X = R_{a_1} \oplus \dots \oplus R_{a_n}$ where R_{a_i} is an indecomposable from the tube $\mathcal{T}_{a_i}$, a_i are pairwise different and $\text{top}R_{a_i} = R_P(1, a_i)$. Moreover $\underline{\dim}X = \underline{\dim}I + \underline{\dim}P$.

We have a short exact sequence $0 \to R'_{a_i} \overset{u}{\hookrightarrow} R_{a_i} \to R_P(1, a_i) \to 0$ which induces the exact sequence $0 \to \text{Hom}(P, R'_{a_i}) \to \text{Hom}(P, R_{a_i}) \to \text{Hom}(P, R_P(1, a_i)) \to 0$. Since $\dim_k \text{Hom}(P, R_P(1, a_i)) = \deg a_i$, $\text{Hom}(P, u)$ is not an isomorphism. It follows that there is f_i nonzero in $\text{Hom}(P, R_{a_i})$ such that $\forall g_i \in \text{Hom}(P, R'_{a_i})$ we have $ug_i \neq f_i$. So f_i does not factor through any proper regular submodule of R_{a_i}.

Note that if $\underline{\dim}P < \underline{\dim}R_{a_i}$, then f_i is a monomorphism. Indeed, suppose that f_i is not a monomorphism. Then by Lemma 1.5 $\text{Im} f_i$ is regular; but $\text{Im} f_i \subset R_{a_i}$, so f_i would factor through $\text{Im} f_i$, which is a contradiction.

If $\underline{\dim}P \nless \underline{\dim}R_{a_i}$, then again by Lemma 1.5 $\text{Im} f_i$ is regular and f_i factors through $\text{Im} f_i$. So $\text{Im} f_i = R_{a_i}$ which means that f_i is an epimorphism and in this way $\underline{\dim}P > \underline{\dim}R_{a_i}$.

Now consider the morphism $f = (f_i) : P \to R_{a_1} \oplus \dots \oplus R_{a_n}$. If one f_i is a monomorphism, then f is trivially a monomorphism. If all f_i are epimorphisms and f is not a monomorphism, then by Lemma 1.5 $\text{Im} f \subseteq R_{a_1} \oplus \dots \oplus R_{a_n}$ is regular, so $\text{Im} f = R'_{a_1} \oplus \dots \oplus R'_{a_n}$ with $R'_{a_i} \subseteq R_{a_i}$. But $f_i = p_i f$ is an epimorphism, so $R'_{a_i} = R_{a_i}$ which means that f is an epimorphism contradicting Condition (i). We conclude that f is a monomorphism.

By Lemma 1.4 and looking at the defects, $\text{Coker} f$ is either an indecomposable preinjective or the direct sum of an indecomposable preinjective and a regular module with components from the tubes $\mathcal{T}_{x_i}$. We show that this second case is impossible. Indeed, suppose without loss of generality that

Coker $f = X \oplus R''_{a_1}$ where R''_{a_1} is an indecomposable from the tube $\mathcal{T}_{a_1}$. So we have the exact sequence

$$0 \to P \xrightarrow{f=\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}} R_{a_1} \oplus \ldots \oplus R_{a_n} \xrightarrow{\begin{pmatrix} g_1 & \ldots & g_n \\ h_1 & \ldots & h_n \end{pmatrix}} X \oplus R''_{a_1} \to 0.$$

Note that $h_2 = 0$, ..., $h_n = 0$ so $h_1 : R_{a_1} \to R''_{a_1}$ is an epimorphism and $h_1 f_1 = 0$. It follows that $0 \neq \operatorname{Im} f_1 \subseteq \operatorname{Ker} h_1$ and $\operatorname{Ker} h_1$ is a proper regular submodule of R_{a_1}. So f_1 factors through a proper regular submodule of R_{a_1}, which is a contradiction. We conclude that Coker f is an indecomposable preinjective and since $\underline{\dim}\operatorname{Coker} f = \underline{\dim} I$, we have that Coker $f \cong I$.

Second step. Using the previous step, we have an exact sequence $0 \to P \to X \to I \to 0$ with X satisfying condition list (2). It follows that $\underline{\dim} I + \underline{\dim} P = \underline{\dim} X = d'\delta + \sum_{e \in \mathcal{S}} \sigma'_e$ where either $\sigma_e = 0$, or else $0 < \sigma_e$ is the dimension of an indecomposable regular R_e from the non-homogeneous tube $\mathcal{T}_e$ with quasi-top $R_P(1,e)$. Suppose $\sigma'_e = d_e\delta + \sigma_e$ with $0 \leq \sigma_e < \delta$ and $0 \leq d_e$. If $d_e \neq 0$ then there is a unique indecomposable regular R_{d_e} of dimension $d_e\delta$ from the non-homogeneous tube $\mathcal{T}_e$ which embeds into R_e. The factor will be of dimension σ_e with top $R_P(1,e)$ (if $\sigma_e \neq 0$). Let $d = d' + \sum_{e \in \mathcal{S}} d_e$.

We show that $\dim_k \operatorname{Ext}^1(I,P) = d+1$. Suppose first that we don't have non-homogeneous tubes, so we are in the Kronecker case (see Lemma 1.8). In this case $\delta = (1,1)$, $\underline{\dim} I + \underline{\dim} P = d\delta$ and then $\dim_k \operatorname{Ext}^1(I,P) = d+1$.

Consider now the case when we do have non-homogeneous tubes, and suppose $d\delta + \sigma_e \neq 0$ for some $e \in \mathcal{S}$. Then there are unique regular indecomposables $R_e \in \mathcal{T}_e$ of dimension $d\delta + \sigma_e$ and top $R_P(1,e)$ and $R_{e'} \in \mathcal{T}_{e'}$ of dimension $\sigma_{e'}$ and top $R_P(1,e')$ for $e' \in \mathcal{S}'^* = \{e' \in \mathcal{S} | e' \neq e, \sigma_{e'} \neq 0\}$. Suppose that $\mathcal{S}^* = \{e' \in \mathcal{S} | \sigma_{e'} \neq 0\}$ (note that it can be empty). Let $R = R_e \oplus \bigoplus_{e' \in \mathcal{S}'^*} R_{e'}$. It follows from the previous step that $F^R_{IP} \neq 0$, so we have a short exact sequence $0 \to P \to R \to I \to 0$, which induces the exact sequence

$$0 \to \operatorname{Hom}(I,P) \to \operatorname{Hom}(R,P) \to \operatorname{End}(P) \to \operatorname{Ext}^1(I,P) \to \operatorname{Ext}^1(R,P) \to \operatorname{Ext}^1(P,P).$$

We deduce by using Lemma 1.4 that

$$\dim_k \operatorname{Ext}^1(I,P) = 1 + \dim_k \operatorname{Ext}^1(R,P) = 1 - \langle \underline{\dim} R, \underline{\dim} P \rangle = 1 - \langle t_0\delta + \sum_{e' \in \mathcal{S}^*} \sigma_{e'}, \underline{\dim} P \rangle$$

$$= 1 + d - \langle \sum_{e' \in \mathcal{S}^*} \sigma_{e'}, \underline{\dim} P \rangle,$$

so we need to prove that for $e' \in \mathcal{S}^*$ we have $\langle \sigma_{e'}, \underline{\dim} P \rangle = 0$. Indeed, if m is the rank of the tube $\mathcal{T}_{e'}$ then

$$1 = -\langle \delta, \underline{\dim} P \rangle = -\sum_{i=1}^{m} \langle \underline{\dim}\, {}^{i}R(1,e'), \underline{\dim} P \rangle = \sum_{i=1}^{m} \dim_k \operatorname{Ext}^1({}^{i}R(1,e'), P),$$

so there is a unique $i_0 \in \{1,\ldots,m\}$ with $\dim_k \operatorname{Ext}^1({}^{i_0}R(1,e'),P) \neq 0$. Using the Auslander-Reiten formulas we can see that $\dim_k \operatorname{Ext}^1({}^{i_0}R(1,e'),P) = \dim_k \operatorname{Hom}(P, \tau\, {}^{i_0}R(1,e'))$, so ${}^{i_0}R(1,e') =$

$\tau^{-1}R_P(1,e')$. In this way if $1 \le u < m$ is the quasi-length corresponding to $\sigma_{e'}$, then

$$\begin{aligned}\langle \sigma_{e'}, \underline{\dim} P\rangle &= \langle \underline{\dim}\tau^{u-1}R_P(1,e') + \cdots + \underline{\dim} R_P(1,e'), \underline{\dim} P\rangle \\ &= -\dim_k \mathrm{Ext}^1(\tau^{u-1}R_P(1,e'),P) - \cdots - \dim_k \mathrm{Ext}^1(R_P(1,e'),P) = 0.\end{aligned}$$

Third step. We know from the previous steps that $\dim_k \mathrm{Ext}^1(I,P) = d+1$ and

$$|\mathrm{Ext}^1(I,P)| - 1 = \sum_{\substack{[X] \\ X \text{ satisfying condition list (2)}}} |\mathrm{Ext}^1(I,P)_X|,$$

so

$$q^{d+1} - 1 = \sum_{\substack{[X] \\ X \text{ satisfying condition list (2)}}} |\mathrm{Ext}^1(I,P)_X|$$

where the terms in the last sum are at least $q-1$ (see Lemma 1.6 (a)).

Now we will count the number of nonisomorphic regulars satisfying condition list (2). As in the previous subsection, for $\mathcal{T}_a$ a homogeneous tube and $t \ge 1$ denote by $R_a(t)$ the regular indecomposable $R(t,a)$ and let $R_a(0) = 0$. For $\mathcal{T}_e$ ($e \in \mathcal{S}$) a non-homogeneous tube and $t \ne 0$ denote by $R_e(t)$ the unique indecomposable from $\mathcal{T}_e$ of dimension $t\delta + \sigma_e$ with top $R_P(1,e)$. For $t = 0$ and $\sigma^e \ne 0$ let $R_e(0)$ be the unique indecomposable from $\mathcal{T}_e$ of dimension σ_e with top $R_P(1,e)$. For $t = 0$ and $\sigma_e = 0$ let $R_e(0) = 0$. Then

$$\bigoplus_{\substack{(t_a)_{a \text{ closed point in } \mathbf{P}^1(k)} \\ t_a \in \mathbb{Z}, t_a \ge 0 \\ \sum_a t_a \deg a = d}} R_a(t_a)$$

are nonisomorphic regulars satisfying condition list (2) so by Lemma 1.6 we have exactly $\frac{q^{d+1}-1}{q-1}$ of them. It follows that for each X satisfying condition list (2) we have $|\mathrm{Ext}^1(I,P)_X| = q-1$. Finally, applying Riedtmann's formula we get for X of good type

$$F_{IP}^X = \frac{\alpha_X |\mathrm{Ext}^1(I,P)_X|}{\alpha_I \alpha_P |\mathrm{Hom}(I,P)|} = \frac{\alpha_X}{q-1}.$$

□

Specializing the proposition above to indecomposables we obtain:

Corollary 2.2. *For positive real roots p, i with $\partial p = -1$, $\partial i = 1$ (thus corresponding to an indecomposable preprojective P and preinjective I) we have that $F_{ip}^r = \frac{1}{q-1}a_r(q)$ for any symbol r corresponding to regular indecomposables of dimension $p+i$ taken from a tube $\mathcal{T}_a$ and having as regular top $R_P(1,a)$, where $a_r(q)$ is the number of automorphisms of any regular corresponding to the symbol r (see Lemma 1.7). Moreover, such a symbol r exists if and only if $\langle i,p\rangle < 0$. For any other symbol α we have $F_{ip}^\alpha = 0$.*

The previous proposition leads us to a generalization of Lemma 17 from [26].

Corollary 2.3. *Let X be a regular module, I an indecomposable preinjective of defect 1 with $\underline{\dim} I > n\delta$ and P an indecomposable preprojective of defect -1. Then $F_{IP}^X = F_{I'P'}^X$ where I', P' are the indecompos-*

ables with $\underline{\dim} I - \underline{\dim} I' = n\delta = \underline{\dim} P' - \underline{\dim} P$. *(Note that* I', P' *are uniquely determined,* $\partial I' = 1$ *and* $\partial P' = -1$*).*

Proof. Note that X satisfies condition list (2) relatively to I and P if and only if it satisfies condition list (2) relatively to I' and P'. Indeed, $\underline{\dim} I + \underline{\dim} P = \underline{\dim} I' + \underline{\dim} P'$ and $R_P(1,a) = R_{P'}(1,a)$, because $\dim_k \operatorname{Hom}(P, R(1,a)) = \langle \underline{\dim} P, \underline{\dim} R(1,a)\rangle = \langle \underline{\dim} P + n\delta, \underline{\dim} R(1,a)\rangle = \langle \underline{\dim} P', \underline{\dim} R(1,a)\rangle = \dim_k \operatorname{Hom}(P', R(1,a))$. Now the assertion follows from the previous proposition. □

2.1.5 Some Ringel-Hall products

We will present some particular Ringel-Hall products involving modules with indecomposable components of absolute defect up to 1.

Our first formula deals with the Ringel-Hall product $[R][P]$ where R is a regular with indecomposable components from the same homogeneous tube and P is an indecomposable preprojective of defect -1. Dually, we can easily derive a formula for $[I][R]$ where I is an indecomposable preinjective of defect 1.

The second formula describes the product $[I][P]$, where I is a preinjective indecomposable with $\partial I = 1$ and P is a preprojective indecomposable with $\partial P = -1$.

Proposition 2.4. *We have the following:*

(a) For an indecomposable preprojective P *of defect* -1*, a homogeneous tube* $\mathcal{T}_a$ *and a partition* λ

$$[R(\lambda,a)][P] = \sum q^{|\mu|\deg a} \cdot g^{\lambda}_{\mu(|\lambda-\mu|)}(q^{\deg a}) \cdot \frac{\alpha_{R(|\lambda-\mu|,a)}\alpha_{R(\mu,a)}}{\alpha_{R(\lambda,a)}} [P(+(|\lambda|-|\mu|)\delta \deg a) \oplus R(\mu,a)],$$

the summation going over all partitions μ *such that* $\lambda - \mu$ *is a horizontal strip. Here* g *denotes the classical Hall polynomial and* $P(+(|\lambda|-|\mu|)\delta\deg a)$ *is the indecomposable preprojective of dimension* $\underline{\dim} P + (|\lambda|-|\mu|)\delta\deg a$*.*

In particular

$$[tR(1,a)][P] = q^{t\deg a}[P \oplus tR(1,a)] + [P(+\delta\deg a) \oplus (t-1)R(1,a)],$$

where $P(+\delta\deg a)$ *denotes the indecomposable preprojective of dimension* $\underline{\dim} P + \delta\deg a$*.*

We also have

$$[R(t,a)][P] = q^{t\deg a}[P \oplus R(t,a)] + [P(+t\delta\deg a)] + \\ + \sum_{i=1}^{t-1}(q^{(t-i)\deg a} - q^{(t-i-1)\deg a})[P(+i\delta\deg a) \oplus R(t-i,a)],$$

where $P(+t\delta\deg a)$ *denotes the indecomposable preprojective of dimension* $\underline{\dim} P + t\delta\deg a$ *and* $P(+i\delta\deg a)$ *denotes the indecomposable preprojective of dimension* $\underline{\dim} P + i\delta\deg a$*.*

The formulas above dualize for I *preinjective of defect* 1*.*

(b) If $\langle \underline{\dim} I, \underline{\dim} P\rangle = 0$ *then* $[I][P] = q^{\langle \underline{\dim} P, \underline{\dim} I\rangle}[P \oplus I]$.

If $\langle \underline{\dim} I, \underline{\dim} P\rangle \neq 0$ *then* $[I][P] = q^{\langle \underline{\dim} P, \underline{\dim} I\rangle}[P \oplus I] + \frac{1}{q-1}\sum_{[X]} \alpha_X[X]$ *where the (nonempty) sum is taken over all modules* X *satisfying the condition list (2).*

Proof. (a) Suppose that $F^X_{R(\lambda,a)P} \neq 0$, so we have a short exact sequence $0 \to P \to X \xrightarrow{g} R(\lambda,a) \to 0$. Note that we can't have preinjective components in X (since they would be direct summands in $\operatorname{Ker} g \cong P$), $\partial X = -1$, so due to Lemma 1.4 and Lemma 1.5, X is of the form $X = P(+(|\lambda|-|\mu|)\delta \deg a) \oplus R(\mu,a)$. Here μ is a partition with $|\mu| \leq |\lambda|$, since $\underline{\dim} P \leq \underline{\dim} P(+(|\lambda|-|\mu|)\delta \deg a)$.

If $\mu = (0)$, then by Proposition 2.2 $\lambda = (t)$, so $\lambda - \mu$ is a horizontal t-strip (see Section 1.2) and $F^{P(+(|\lambda|-|\mu|)\delta \deg a)}_{R(t,a)P} = 1 = q^{|\mu| \deg a} \cdot g^{\lambda}_{\mu(|\lambda-\mu|)}(q^{\deg a}) \cdot \frac{\alpha_{R(|\lambda-\mu|,a)}\alpha_{R(\mu,a)}}{\alpha_{R(\lambda,a)}}$.

If $\mu \neq (0)$, then we apply Green's formula (see Proposition 1.3 (c)) with choices $N_1 = R(\lambda,a)$, $N_2 = P$, $N_1' = P(+(|\lambda|-|\mu|)\delta \deg a)$ and $N_2' = R(\mu,a)$. Note that $F^M_{P(+(|\lambda|-|\mu|)\delta \deg a)R(\mu,a)} = 1$ for $M \cong P(+(|\lambda|-|\mu|)\delta \deg a) \oplus R(\mu,a)$ and otherwise is 0 by Riedtmann's formula (see Proposition 1.3 (a)). So using Lemma 1.4, the left hand side of Green's formula becomes

$$\frac{(q-1)}{q^{|\mu| \deg a}} \alpha_{R(\lambda,a)} F^{P(+(|\lambda|-|\mu|)\delta \deg a) \oplus R(\mu,a)}_{R(\lambda,a)P}.$$

Looking at the right hand side of the formula, if the product $F^{R(\lambda,a)}_{RS} F^{P(+(|\lambda|-|\mu|)\delta \deg a)}_{RS'} F^P_{S'T} F^{R(\mu,a)}_{ST}$ is nonzero, then by Lemma 1.4 S', T are either preprojectives or 0. Note that S', T can't be both preprojective since $F^P_{S'T}$ is nonzero and $\partial P = -1$. Also, if $S' = 0$, then $R = P(+(|\lambda|-|\mu|)\delta \deg a)$, so $F^{R(\lambda,a)}_{P(+(|\lambda|-|\mu|)\delta \deg a)S} = 0$. This means that we must have $T = 0$, so $S' = P$, $S = R(\mu,a)$ and using Proposition 2.2 it follows that $R = R(|\lambda-\mu|,a)$. So the right hand side of the formula becomes

$$(q-1)\alpha_{R(|\lambda-\mu|,a)}\alpha_{R(\mu,a)} F^{R(\lambda,a)}_{R(|\lambda-\mu|,a)R(\mu,a)} F^{P(+(|\lambda|-|\mu|,a)\delta \deg a)}_{R(|\lambda-\mu|,a)P}.$$

Note that $\operatorname{Hom}(P, P(+(|\lambda|-|\mu|)\delta \deg a)) \neq 0$ because $\langle \underline{\dim} P, \underline{\dim} P(+(|\lambda|-|\mu|)\delta \deg a)\rangle = \langle \underline{\dim} P, \underline{\dim} P + (|\lambda|-|\mu|)\delta \deg a\rangle = 1 + (|\lambda|-|\mu|)\deg a > 0$. It follows by Proposition 2.2 that $F^{P(+(|\lambda|-|\mu|)\delta \deg a)}_{R(|\lambda-\mu|,a)P} = 1$. On the other hand $F^{R(\lambda,a)}_{R(|\lambda-\mu|,a)R(\mu,a)} = g^{\lambda}_{(|\lambda-\mu|)\mu}(q^{\deg a}) = g^{\lambda}_{\mu(|\lambda-\mu|)}(q^{\deg a})$, so it is 0 unless $\lambda - \mu$ is a horizontal strip (see Section 1.7). Equating the two sides of Green's formula the assertion follows.

(b) Trivial from Proposition 2.3.

□

The third formula describes the Ringel-Hall product $[R][P]$ (and dually $[I][R]$) where R is a regular semisimple taken from a non-homogeneous tube and P is an indecomposable preprojective of defect -1 (and I is an indecomposable preinjective of defect 1).

Consider the non-homogeneous tube $\mathcal{T}_e$ of rank m. Recall from Section 1.4 that we have m regular simples in $\mathcal{T}_e$ denoted by ${}^{i}R(1,e)$ $i \in \{1,\dots,m\}$ such that $\tau({}^{i}R(1,e)) = {}^{i-1}R(1,e)$ for $i \geq 2$, $\tau({}^{1}R(1,e)) = {}^{m}R(1,e)$ and $\sum_{i=1}^{m} \underline{\dim}\, {}^{i}R(1,e) = \delta$. Here ${}^{i}R(t,e)$ will denote the indecomposable regular from $\mathcal{T}_e$ with regular socle ${}^{i}R(1,e)$ and regular length t. A module with all its indecomposable components regular simples from $\mathcal{T}_e$ will be called regular semisimple from $\mathcal{T}_e$. A regular module from the tube $\mathcal{T}_e$ (i.e. a module with all its indecomposable components from $\mathcal{T}_e$) will be denoted by R_e. The isomorphism classes $[R_e]$ of regulars from the tube $\mathcal{T}_e$ and $[0]$ form a $\mathbb{Q}$-basis of a unital $\mathbb{Q}$-subalgebra $\mathcal{H}(\mathcal{T}_e)$ of $\mathcal{H}(kQ)$, called the *Ringel-Hall algebra of the tube* $\mathcal{T}_e$. We know due to Guo (see [25]) the following:

Proposition 2.5. ([25]) *$\mathcal{H}(\mathcal{T}_e)$ is generated by the isoclasses of regular semisimples from $\mathcal{T}_e$.*

This is why we describe the formula $[R_e^s][P]$ with R_e^s a regular semisimple taken from $\mathcal{T}_e$ and P an indecomposable preprojective of defect -1.

We need the following lemma:

Lemma 2.1. *(a) Suppose that $F^{R_e^s}_{R'_e R''_e} \neq 0$, where R'_e, R''_e are regulars from $\mathcal{T}_e$ and R_e^s is a regular semisimple from $\mathcal{T}_e$. Then R'_e and R''_e must be regular semisimples.*

(b) Suppose that $F^{P'}_{R_e^s P} \neq 0$, where R_e^s is a regular semisimple from $\mathcal{T}_e$ and P, P' are indecomposable preprojectives of defect -1. Then $R_e^s = \tau^{-1}R_P(1,e)$ (so it is regular simple). Conversely, for P an indecomposable preprojective of defect -1 there is up to isomorphism a unique indecomposable preprojective P' of dimension $\underline{\dim}P' = \underline{\dim}P + \underline{\dim}\tau^{-1}R_P(1,e)$ and we have $F^{P'}_{\tau^{-1}R_P(1,e)\ P} = 1$.

Proof. (a) Regular modules from $\mathcal{T}_e$ form an extension-closed abelian subcategory of mod-kQ, the simple objects in this subcategory being the regular simple modules. It is known (see [1]) that this category is equivalent to the category of modules of finite length over a specific $m \times m$ matrix ring. So regular submodules and regular factor modules of a regular semisimple module are again regular semisimple.

(b) By Proposition 2.2, $R_e^s = R_{P'}(1,e)$ and $\underline{\dim}P' = \underline{\dim}P + \underline{\dim}R_e^s$. On the other hand $1 = q(\underline{\dim}P') = q(\underline{\dim}P) + \langle \underline{\dim}P, \underline{\dim}R_e^s\rangle + \langle \underline{\dim}R_e^s, \underline{\dim}P\rangle + q(\underline{\dim}R_e^s) = 2 + \dim_k \operatorname{Hom}(P, R_e^s) - \dim_k \operatorname{Ext}^1(R_e^s, P)$. It follows that $\operatorname{Ext}^1(R_e^s, P) \neq 0$, so using the same argument as at the end of the second step of the proof of Proposition 2.3 (c), we have $R_e^s = R_{P'}(1,e) = \tau^{-1}R_P(1,e)$. Conversely, one can easily see as above that $q(\underline{\dim}P + \underline{\dim}\tau^{-1}R_P(1,e)) = 1$ so there is up to isomorphism a unique indecomposable preprojective P' of dimension $\underline{\dim}P' = \underline{\dim}P + \underline{\dim}\tau^{-1}R_P(1,e)$. Note that $\langle \underline{\dim}P, \underline{\dim}P'\rangle = q(\underline{\dim}P) + \dim_k \operatorname{Hom}(P, \tau^{-1}R_P(1,e)) = 1 \neq 0$ so $\operatorname{Hom}(P,P') \neq 0$. Since $\langle \underline{\dim}P', \underline{\dim}\tau^{-1}R_P(1,e)\rangle = 1 \neq 0$ it follows that $R_{P'}(1,e) = \tau^{-1}R_P(1,e)$, so we are done using Proposition 2.2.

□

Theorem 2.4. *Let $R_e^s = \oplus_{i=1}^m t_i{}^iR(1,e)$ be a regular semisimple from the tube $\mathcal{T}_e$ and P an indecomposable preprojective of defect -1. Suppose that $\tau^{-1}R_P(1,e) = {}^{i_0}R(1,e)$, so $R_P(1,e) = {}^{i_0-1}R(1,e)$ and let $t_0 = t_m$. Then we have:*

(a) If $t_{i_0} = 0$ then $[R_e^s][P] = q^{t_{i_0-1}}[P \oplus R_e^s]$.

(b) If $t_{i_0} > 0$ then $[R_e^s][P] = q^{t_{i_0-1}}[P \oplus R_e^s] + [P' \oplus R'^s_e]$, where P' denotes the (up to isomorphism) unique indecomposable preprojective of dimension $\underline{\dim}P' = \underline{\dim}P + \underline{\dim}{}^{i_0}R(1,e)$ and $R'^s_e = t_1{}^1R(1,e) \oplus \ldots \oplus (t_{i_0} - 1){}^{i_0}R(1,e) \oplus \ldots \oplus t_m{}^mR(1,e)$.

Proof. Suppose that $F^X_{R_e^s P} \neq 0$, so we have a short exact sequence $0 \to P \to X \xrightarrow{g} R_e^s \to 0$. Note that we can't have preinjective components in X (since they would be direct summands in $\operatorname{Ker} g \cong P$), $\partial X = -1$, so due to Lemma 1.4 and Lemma 1.5 X is of the form $X = P' \oplus R_e$. Here P' is an indecomposable preprojective of defect -1 and R_e is a regular module from the tube $\mathcal{T}_e$.

If $R_e = 0$, then by the previous lemma $R_e^s = \tau^{-1}R_P(1,e)$, P' is the indecomposable preprojective of dimension $\underline{\dim}P' = \underline{\dim}P + \underline{\dim}\tau^{-1}R_P(1,e)$ and we have $F^{P'}_{\tau^{-1}R_P(1,e)\ P} = 1$.

If $R_e \neq 0$, then we apply Green's formula (Proposition 1.3 (c)) with choices $N_1 = R_e^s$, $N_2 = P$, $N_1' = P'$ and $N_2' = R_e$. Note that by Riedtmann's formula (Proposition 1.3 (a)) $F^{M}_{P'R_e} = 1$ for $M \cong P' \oplus R_e$ and otherwise is 0. So, using Lemma 1.4, the left hand side of Green's formula becomes

$$\frac{(q-1)}{q^{\dim_k \mathrm{Hom}(P',R_e)}}\alpha_{R_e^s}F^{P'\oplus R_e}_{R_e^s P}.$$

Looking at the right hand side of the formula, if the product $F^{R_e^s}_{RS}F^{P'}_{RS'}F^{P}_{S'T}F^{R_e}_{ST}$ is nonzero, then by Lemma 1.4 S', T are either preprojectives or 0. Note that S', T can't be both preprojective since $F^{P}_{S'T}$ is nonzero and $\partial P = -1$. Also, if $S' = 0$, then $R = P'$, so $F^{R_e^s}_{P'S} = 0$. This means that we must have $T = 0$, so $S' = P$, $S = R_e$. By the previous lemma we can have $R = 0$ or $R = \tau^{-1}R_P(1,e) = {}^{i_0}R(1,e)$. In the case $R = 0$ we have $P' = P$, $R_e^s = R_e$, so the right hand side of the formula becomes

$$(q-1)\alpha_{R_e^s}$$

and thus $F^{P\oplus R_e^s}_{R_e^s P} = q^{\dim_k \mathrm{Hom}(P,R_e^s)} = q^{t_{i_0}-1}$ (where $t_0 = t_m$). Next, we consider the case $R = {}^{i_0}R(1,e)$. The right hand side of the formula in this case is

$$(q-1)\alpha_{{}^{i_0}R(1,e)}\alpha_{R_e}F^{R_e^s}_{{}^{i_0}R(1,e)R_e}F^{P'}_{{}^{i_0}R(1,e)P}.$$

Here P' is the indecomposable preprojective of dimension $\underline{\dim}P' = \underline{\dim}P + \underline{\dim}{}^{i_0}R(1,e)$ and we have $F^{P'}_{{}^{i_0}R(1,e)P} = 1$. On the other hand one can easily see that $F^{R_e^s}_{{}^{i_0}R(1,e)R_e} \neq 0$ implies $t_{i_0} \geq 1$ and $R_e = R_e'^s = t_1{}^1R(1,e) \oplus \ldots \oplus (t_{i_0}-1)^{i_0}R(1,e) \oplus \ldots \oplus t_m{}^mR(1,e)$, so $F^{R_e^s}_{{}^{i_0}R(1,e)R_e'^s} = \frac{q^{t_{i_0}}-1}{q-1}$. Equating the two sides of Green's formula and using our knowledge on the number of automorphisms (see Lemma 1.4) we get $F^{P'\oplus R_e'^s}_{R_e^s P} = 1$. □

2.2 The Ringel-Hall polynomials $F^{\delta}_{\delta-pp}$, where p is a positive real root of arbitrary negative defect

Let $x < \delta$ be a positive real root of arbitrary negative defect and consider the Ringel-Hall numbers of the form $F^{R(1,a)}_{I(\delta-x)P(x)}$, where $R(1,a)$ is a homogeneous regular of dimension δ (thus a regular simple), $P(x)$ is the unique indecomposable preprojective with $\underline{\dim}P(x) = x$ and $I(\delta-x)$ is the unique indecomposable preinjective with $\underline{\dim}I = \delta - x$ of defect $-\partial x$. Remember that in the $\widetilde{\mathbb{A}}_m$ case over a field with 2 elements we don't have homogeneous regulars of dimension δ.

Let $R(1,a)$ and $R(1,a')$ be two homogeneous modules of dimension δ. Then by Theorem 1.1 we have a Ringel-Hall polynomial $F^{\delta}_{\delta-x\,x}$ such that

$$F^{R(1,a)}_{I(\delta-x)P(x)} = F^{R(1,a')}_{I(\delta-x)P(x)} = F^{\delta}_{\delta-x\,x}(q).$$

Let $e_i = (0,\dots,1,\dots 0)$ (with 1 at the i-th place) be the dimension of the simple projective module corresponding to the unique sink i in Q_i. Using the notions and results from Section 1.9 we show that in case Q is a tree (thus not of type $\widetilde{\mathbb{A}}_m$), the Ringel-Hall polynomials $F^{\delta}_{\delta-x\,x}$ over the quiver Q are equal to special Ringel-Hall polynomials $F^{\delta}_{\delta-e_i\,e_i}$ over unique sink quivers Q_i.

Proposition 2.6. *There is a vertex i (with $\delta_i = -\partial x$) such that ${}^{Q}F^{\delta}_{\delta-x\,x} = {}^{Q_i}F^{\delta}_{\delta-e_i\,e_i}$, where the first polynomial is taken over the quiver Q and the second one over the quiver Q_i.*

Proof. We know that there exists field independently a sequence $i_1,\dots,i_t$ of vertices in Q such that for each $s \in \{1,\dots,t\}$ the vertex i_s is a sink in $\sigma_{i_{s-1}}\dots\sigma_{i_1}Q$ and $S^+_{i_t}\dots S^+_{i_1}P(x) = S''(i) \in$ mod-kQ'' is a simple projective corresponding to the sink i in $Q'' = \sigma_{i_t}\dots\sigma_{i_1}Q$. It follows that

$$F^{R(1,a)}_{I(\delta-x)P(x)} = F^{S^+_{i_t}\dots S^+_{i_1}R(1,a)}_{S^+_{i_s}\dots S^+_{i_1}I(\delta-x)\;S''(i)} = F^{R''}_{S^+_{i_s}\dots S^+_{i_1}I(\delta-x)\;S''(i)},$$

where R'' is a homogeneous regular module in mod-kQ'' of dimension δ. Also, there is a sequence $j_1,\dots,j_r$ of vertices in Q'' different from i and not in N_i (the set of neighbors of i) such that for each $s \in \{1,\dots,t\}$ the vertex j_s is a sink in $\sigma_{j_{s-1}}\dots\sigma_{j_1}Q''$ and $\sigma_{j_r}\dots\sigma_{j_1}Q'' = Q_i$. It follows that $S^+_{j_r}\dots S^+_{j_1}S''(i) = S'(i)$ is the simple projective in mod-kQ_i corresponding to the unique sink i in Q_i. The statement now follows using the same argument as above. ☐

Using a computer program written in GAP (see [23]) we have computed the special Ringel-Hall polynomials from above. The program computes the Ringel-Hall numbers over small finite fields and interpolates the Ringel-Hall polynomials. Due to the particular orientation of Q_i (to which all the other orientations are reduced via the previous proposition), the low dimensions and the symmetries, only a few cases occur and thus the computing time is very short. It takes around 15 minutes to obtain the polynomial list in the following theorem. We should also remark that using our program we could reproduce Ringel's list of Ringel-Hall polynomials for the simply-laced Dynkin case (see [42]).

In case Q is not a tree, thus of type $\widetilde{\mathbb{A}}_m$, we must have $\partial x = -1$ and then by Proposition 2.3 and Lemma 1.4 we have that $F^{\delta}_{\delta-x\,x}(q) = \frac{1}{q-1}\alpha_{R(1,a)} = 1$.

Summarizing all of the above we obtain:

Theorem 2.5 ([57]). *Let x be a positive real root with $\partial x < 0$. Then $F^{\delta}_{\delta-x\,x} = f_{-\partial x}$, where:*

$$\begin{aligned}
f_1 &= 1,\\
f_2 &= X - 3,\\
f_3 &= X^2 - 5X + 7,\\
f_4 &= X^3 - 6X^2 + 15X - 14,\\
f_5 &= X^4 - 7X^3 + 22X^2 - 37X + 26,\\
f_6 &= X^5 - 7X^4 + 22X^3 - 45X^2 + 62X - 39.
\end{aligned}$$

2.3 Ringel-Hall polynomials involving indecomposable modules of absolute defect up to 2

2.3.1 Morphisms from a preprojective indecomposable of defect -2 to regular non-homogeneous simples

We begin with a technical lemma on morphisms from a preprojective indecomposable of defect −2 to regular non-homogeneous simples, which will be needed later. Using the notations from Proposition 2.2, we get:

Lemma 2.2. *(a) Let P be an indecomposable preprojective module of defect -2 and $\mathcal{T}_e$ a non-homogeneous tube. Then we have exactly two regular-simples $[R_P^1(1,e)] \neq [R_P^2(1,e)]$ on the mouth of the tube such that $\dim_k \mathrm{Hom}(P, R_P^1(1,e)) = \langle \underline{\dim} P, \underline{\dim} R_P^1(1,e)\rangle = \langle \underline{\dim} P, \underline{\dim} R_P^2(1,e)\rangle = \dim_k \mathrm{Hom}(P, R_P^2(1,e)) = 1$.*

(b) Suppose $\underline{\dim} P = \underline{\dim} P_0 + \underline{\dim} P_1$, $\underline{\dim} P = \underline{\dim} P' + \underline{\dim} P''$ and $\underline{\dim} P'' - \underline{\dim} P_0 = \underline{\dim} P_1 - \underline{\dim} P' = d\delta + \sigma_0 + \sigma_1 + \sigma_\infty$, where all the modules are preprojective indecomposable with $\partial P_0 = \partial P_1 = \partial P' = \partial P'' = -1$, $\partial P = -2$, $0 < \langle \underline{\dim} P_0, \underline{\dim} P''\rangle = \langle \underline{\dim} P', \underline{\dim} P_1\rangle = d+1$ and $0 \leq \sigma_e < \delta$ if nonzero is the dimension of a unique regular non-homogeneous indecomposable from $\mathcal{T}_e$ (thus a root). Then $[R_{P''}(1,e)] = [R_{P_1}(1,e)]$ if and only if $\sigma_e \neq 0$ and for $\mathcal{T}_a$ homogeneous $[R_{P''}(1,a)] = [R_{P_1}(1,a)]$.

Proof. (a) Since $-\partial P = 2 = \langle \underline{\dim} P, \delta\rangle = \langle \underline{\dim} P, \sum_{i=1}^{m} \underline{\dim}\, {}^i R(1,e)\rangle$ it follows that $\langle \underline{\dim} P, \underline{\dim}\, {}^i R(1,e)\rangle \in \{0,1,2\}$. Using a convenient orientation for each quiver, the statement follows via direct computation.

(b) We only have to look at the non-homogeneous case. Suppose that $[R_{P''}(1,e)] = [R_{P_1}(1,e)]$ and $\sigma_e = 0$. So by (a) (without loss of generality) $[R_{P''}(1,e)] = [R_{P_1}(1,e)] = [R_P^1(1,e)]$ and $[R_{P'}(1,e)] = [R_{P_0}(1,e)] = [R_P^2(1,e)]$. But this means

$$\begin{aligned}0 &= \langle d\delta, \underline{\dim} R_P^1(1,e)\rangle = \langle d\delta + \sigma_e, \underline{\dim} R_P^1(1,e)\rangle = \langle d\delta + \sigma_0 + \sigma_1 + \sigma_\infty, \underline{\dim} R_P^1(1,e)\rangle \\ &= \langle \underline{\dim} P_1 - \underline{\dim} P', \underline{\dim} R_P^1(1,e)\rangle,\end{aligned}$$

so $1 = \langle \underline{\dim} P_1, \underline{\dim} R_P^1(1,e)\rangle = \langle \underline{\dim} P', \underline{\dim} R_P^1(1,e)\rangle = 0$, a contradiction.

Suppose that $\sigma_e \neq 0$. Using Proposition 2.2, we know that the regular top of the regular non-homogeneous of dimension σ_e must be $[R_{P''}(1,e)] = [R_{P_1}(1,e)]$.

□

2.3.2 Reductions and main tools

Our aim is to determine the tame Ringel-Hall polynomials F_{xy}^z with $z = x + y$ and x, y, z corresponding to indecomposable modules. Using the results from Section 2.1, we may suppose that at least one of the defects $\partial x, \partial y, \partial z$ is ± 2. This also means that from now on we may suppose that our tame quiver is a tree (thus not of type $\widetilde{\mathbb{A}}_m$).

We will use the following reductions:

(a) We can choose any orientation for every tame quiver since Ringel-Hall polynomials are reflection independent, up to a simple projective or injective (see the previous section).

(b) Using duality arguments we can interchange preinjectives with preprojectives.

By the reductions above one can see that we may get nonzero F^z_{xy} Ringel-Hall polynomials in the following cases (here all the symbols x, y, z correspond to indecomposables and also $z = x + y$):

(1) $F^p_{p'p_0}$, where $\partial p = -2$, $\partial p_0 = \partial p' = -1$, or dually $F^i_{i_0 i'}$, where $\partial i = 2$, $\partial 1_0 = \partial i' = 1$;

(2) $F^p_{i_0 p'}$, where $\partial p = -1$, $\partial p' = -2$, $\partial i_0 = 1$, or dually $F^i_{i' p_0}$, where $\partial i = 1$, $\partial i' = 2$, $\partial p_0 = -1$;

(3) $F^p_{r p_0}$, where $\partial p = -2$, $\partial p_0 = -2$, r is the symbol of a homogeneous regular, or dually $F^i_{i_0 r}$, where $\partial i = 2$, $\partial i_0 = 2$, r is the symbol of a homogeneous regular;

(4) $F^r_{i_0 p}$, where $\partial p = -2$, $\partial i_0 = 2$ and r is the symbol of a homogeneous regular;

(5) $F^p_{r p_0}$, where $\partial p = -2$, $\partial p_0 = -2$, r is the symbol of a non-homogeneous regular, or dually $F^i_{i_0 r}$, where $\partial i = 2$, $\partial i_0 = 2$, r is the symbol of a non-homogeneous regular;

(6) $F^r_{i_0 p}$, where $\partial p = -2$, $\partial i_0 = 2$ and r is the symbol of a non-homogeneous regular.

We end this section with the main tools used in the computation of the Ringel-Hall numbers above. The first tool is the double, dimension-inductive formula by Ringel for counting the number of mono- and epimorphisms presented in Chapter 1 as Proposition 1.4.

Our second tool is derived from the associativity of the Ringel-Hall algebra and will permit us to "decompose" the indecomposable y of defect -2 from F^z_{xy} via an orthogonal exceptional pair of preprojective indecomposables of defect -1. In this way F^z_{xy} can be obtained using Ringel-Hall polynomials involving only indecomposables of defect ± 1 or 0.

Consider the sums of the form:

$$S_a(X,Y,Z,T) := \sum_{[U]} F^U_{XY} F^Z_{UT},\ S_g(X,Y,Z,T) := \frac{1}{\alpha_X} \sum_{[U]} F^X_{UY} F^Z_{UT} \alpha_U.$$

Proposition 2.7. *Let P be an indecomposable preprojective module and (P'', P') a corresponding orthogonal exceptional pair of indecomposable preprojective modules. Denote by X, Y some arbitrary modules. Then*

$$F^Y_{XP} = \sum_{[U]} F^U_{XP''} F^Y_{UP'} - \sum_{[V]} F^V_{XP'} F^Y_{VP''} = S_a(X, P'', Y, P') - S_a(X, P', Y, P'').$$

Proof. Since (P'', P') is an orthogonal exceptional pair of preprojectives corresponding to P, knowing that k^* acts freely on $\mathrm{Ext}^1(P'', P') \setminus \{0\}$ and $\dim_k \mathrm{Ext}^1(P'', P') = 1$, it follows by Proposition 1.3 (a) and Remark 1.3 that $F^W_{P''P'}$ equals 1 for $[W] = [P]$ and $[W] = [P' \oplus P'']$ and $[W] = 0$ in other cases.

In the same way we obtain that $F^W_{P'P''}$ equals 1 for $[W] = [P' \oplus P'']$ and $[W] = 0$ in other cases. Using this and Proposition 1.3 (b) it follows that

$$\sum_{[U]} F^U_{XP''} F^Y_{UP'} = \sum_{[W]} F^Y_{XW} F^W_{P''P'} = F^Y_{XP} + F^Y_{XP' \oplus P''} = F^Y_{XP} + \sum_{[T]} F^Y_{XT} F^T_{P'P''} = F^Y_{XP} + \sum_{[V]} F^V_{XP'} F^Y_{VP''}.$$

□

Our third tool is derived from the co-associativity of the Ringel-Hall algebra (more precisely from Green's formula) and will permit us to "decompose" the indecomposable z of defect -2 from F^z_{xy} via an orthogonal exceptional pair of preprojective indecomposables of defect -1. The existence of such a pair, as we will see later, will be guaranteed in almost all the cases (see Section 1.10). In this way F^z_{xy} can be deduced using Ringel-Hall polynomials involving only indecomposables of defect ± 1 or 0.

Proposition 2.8. *Let P_0 be a simple projective which embeds in P, which is an indecomposable preprojective module of defect -2. Consider (P'', P') an orthogonal exceptional pair of preprojectives corresponding to P. Denote by X an indecomposable module such that $\underline{\dim} P = \underline{\dim} X + \underline{\dim} P_0$, thus either $\partial P_0 = -1$ and $\partial X = -1$ or $\partial P_0 = -2$ and $\partial X = 0$. Then for $P' \ncong P_0$ we have*

$$F^P_{XP_0} = \frac{1}{\alpha_X} \sum_{[U]} F^X_{UP'} F^{P''}_{UP_0} \alpha_U - \frac{1}{\alpha_X} \sum_{[V]} F^X_{VP''} F^{P'}_{VP_0} \alpha_V = S_g(X, P', P'', P_0) - S_g(X, P'', P', P_0).$$

For $P' \cong P_0$ we have $[X] = [P'']$ and $F^P_{XP_0} = F^P_{P''P'} = 1$.

Proof. Using Riedtmann's formula (see Proposition 1.3 (a)) we obtain that $F^W_{P''P'}$ equals 1 for $[W] = [P]$ or $[W] = [P' \oplus P'']$ and 0 in other cases. Also, $F^W_{P'P''}$ equals 1 for $[W] = [P' \oplus P'']$ and 0 in other cases. Thus $[X] = [P''] \Leftrightarrow [P'] = [P_0]$, and in this case $F^P_{XP_0} = 1$.

Suppose $[X] \neq [P'']$ (thus $[P'] \neq [P_0]$). Using Green's formula (see Proposition 1.3 (c)) for $N_1 = X$, $N_2 = P_0$, $N'_1 = P''$, $N'_2 = P'$, the fact that the pair (P'', P') is orthogonal exceptional and the fact that P_0 is simple projective, we obtain that $[T] = [0]$ (since for $[T] = [P_0]$, we have $[S'] = [0]$, thus $[R] = [P'']$, so X projects on P'' which is impossible if $[X] \neq [P'']$, see Lemma 1.5). Thus, we have:

$$\begin{aligned}
&\alpha_X \alpha_{P_0} \alpha_{P''} \alpha_{P'} (F^P_{XP_0} F^P_{P''P'} \alpha_P^{-1} + F^{P' \oplus P''}_{XP_0} F^{P' \oplus P''}_{P''P'} \alpha^{-1}_{P' \oplus P''}) \\
&= \sum_{[R],[S],[S'],[T]} q^{-\langle \underline{\dim} R, \underline{\dim} T \rangle} F^X_{RS} F^{P''}_{RS'} F^{P_0}_{S'T} F^{P'}_{ST} \alpha_R \alpha_S \alpha_{S'} \alpha_T \\
&= \alpha_{P'} \alpha_{P_0} \sum_{[R]} F^X_{RP'} F^{P''}_{RP_0} \alpha_R,
\end{aligned}$$

thus

$$\alpha_X F^P_{XP_0} + \alpha_X (q-1) F^{P' \oplus P''}_{XP_0} \alpha^{-1}_{P' \oplus P''} = \sum_{[U]} F^X_{UP'} F^{P''}_{UP_0} \alpha_U. \tag{2.1}$$

Using Green's formula for $M = P' \oplus P''$, $N_1 = X$, $N_2 = P_0$, $N'_1 = P'$, $N'_2 = P''$, the fact that the pair (P'', P') is orthogonal exceptional and the fact that P_0 is simple projective, we obtain that $[T] = [0]$ (since for $[T] = [P_0]$, we have $[S'] = [0]$, thus $[R] = [P']$, so X projects on P' which is impossible), see Lemma 1.5. Thus, we have:

$$\begin{aligned}\alpha_X\alpha_{P_0}\alpha_{P'}\alpha_{P''}F^{P'\oplus P''}_{XP_0}\,F^{P'\oplus P''}_{P'P''}\,\alpha^{-1}_{P'\oplus P''}\\ &= \sum_{[R],[S],[S'],[T]} q^{-\langle \underline{\dim}R,\underline{\dim}T\rangle}F^X_{RS}F^{P'}_{RS'}F^{P_0}_{S'T}F^{P''}_{ST}\alpha_R\alpha_S\alpha_{S'}\alpha_T\\ &= \alpha_{P''}\alpha_{P_0}\sum_{[R]}F^X_{RP''}F^{P'}_{RP_0}\alpha_R,\end{aligned}$$

thus

$$\alpha_X(q-1)F^{P'\oplus P''}_{XP_0}\,\alpha^{-1}_{P'\oplus P''} = \sum_{[V]}F^X_{VP'}F^{P''}_{VP_0}\alpha_V. \tag{2.2}$$

The formulas (2.1), (2.2) imply

$$F^P_{XP_0} = \frac{1}{\alpha_X}\left(\sum_{[U]}F^X_{UP'}F^{P''}_{UP_0}\alpha_U - \sum_{[V]}F^X_{VP''}F^{P'}_{VP_0}\alpha_V\right).$$

□

To obtain the listed Ringel-Hall polynomials we will proceed as follows:

(a) In the cases (1), (3), (5) we decompose the indecomposable p of defect -2 via a preprojective orthogonal exceptional pair of defect -1 and use our third tool (co-associativity).

(b) In the cases (2), (4), (6) we may decompose the indecomposable p of defect -2 via a preprojective orthogonal exceptional pair of defect -1 and use our second tool (associativity). However we will follow an easier and quicker approach: after applying a proper reflection functor we will use our first tool (Ringel's formula).

Note that the existence of an orthogonal exceptional preprojective pair of defect -1 will be guaranteed (almost always) by the results in [60] listing all the Schofield pairs corresponding to an exceptional module (see Section 1.10 and Appendix A).

So in fact we follow a particular Schofield induction for the decomposition of a proper indecomposable, combined with our three main tools.

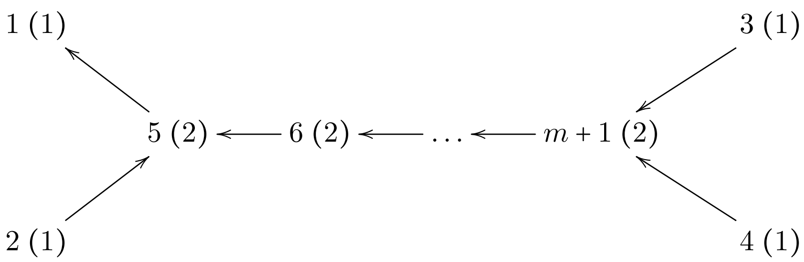
2.3.3 The polynomials $F^p_{p_1p_0}$, where $\partial p = -2$, $\partial p_0 = \partial p_1 = -1$

Theorem 2.6. *Suppose Q is a tame quiver and P, P_0, P_1 are indecomposable preprojectives with defects $\partial P = -2$, $\partial P_0 = \partial P_1 = -1$, such that $\underline{\dim}P = \underline{\dim}P_0 + \underline{\dim}P_1$. Then $F^P_{P_1P_0} = f_{n-1}(q)$, where $n = \langle \underline{\dim}P_0, \underline{\dim}P\rangle = \langle \underline{\dim}P, \underline{\dim}P_1\rangle$ and f_n is the polynomial from Proposition 1.2 (e).*

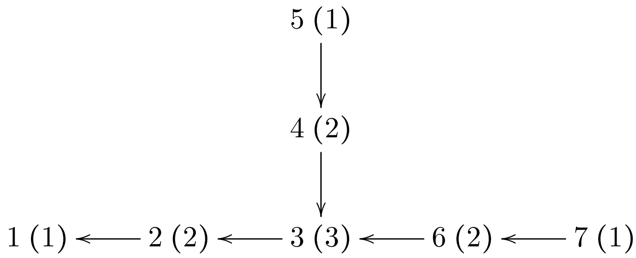
Proof. Trivially $n = \langle \underline{\dim}P_0, \underline{\dim}P\rangle = \langle \underline{\dim}P_0, \underline{\dim}P_0 + \underline{\dim}P_1\rangle = 1 + \langle \underline{\dim}P_0, \underline{\dim}P_1\rangle = \langle \underline{\dim}P_0 + \underline{\dim}P_1, \underline{\dim}P_1\rangle = \langle \underline{\dim}P, \underline{\dim}P_1\rangle$. Note that in case $n \le 0$ – thus in case $\mathrm{Hom}(P_0, P) = 0$ or $\mathrm{Hom}(P, P_1) = 0$ – the Ringel-Hall number $F^P_{P_1P_0} = 0 = f_{n-1}(q)$, so the assertion is true. From now on we will suppose that $n > 0$, thus $\mathrm{Hom}(P_0, P) \neq 0$ and $\mathrm{Hom}(P, P_1) \neq 0$.

Using a succession of reflection functors (corresponding to sinks) and symmetry, we can modify the orientation of Q such that it possesses a unique sink denoted by 1 and P_0 is the simple projective corresponding to this sink, so $n = (\underline{\dim} P)_1$. Note that during this process P and P_1 is never transformed into a simple projective. Indeed, the relation $\underline{\dim} P = \underline{\dim} P_0 + \underline{\dim} P_1$ is kept even after reflections, so P can't be simple and in case P_1 is a simple projective, then since $\mathrm{Hom}(P, P_1) \neq 0$ we get that P_1 is a direct summand of P, a contradiction. Also observe that Ringel-Hall numbers, dimensions of homomorphisms and Euler products are not modified by the reflection functors. So without loss of generality we may suppose that our tame (tree) quiver is one of the following (the vertices are labeled by their numbering and (in parentheses) the corresponding value of the minimal radical vector):

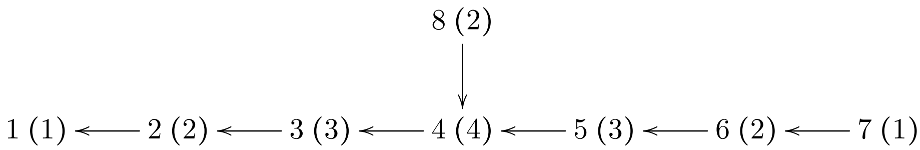
Type $\widetilde{\mathbb{D}}_m$, with $m \geq 4$:



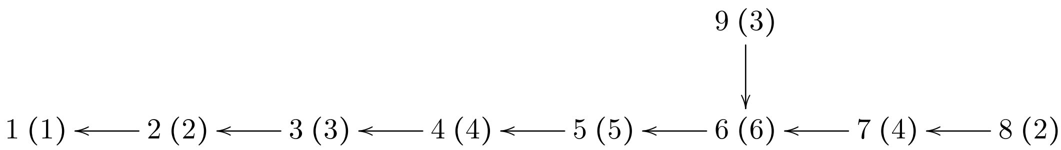
Type $\widetilde{\mathbb{E}}_6$:



Type $\widetilde{\mathbb{E}}_7$:



Type $\widetilde{\mathbb{E}}_8$:



Suppose (P'', P') is a preprojective orthogonal exceptional pair corresponding to P. Such pairs exist (for the given orientation and context) and will be given explicitly in Appendix B.

In case $P_0 \cong P'$ we know that $P_1 \cong P''$ so $\langle \underline{\dim} P_0, \underline{\dim} P\rangle = 1 + \langle \underline{\dim} P_0, \underline{\dim} P_1\rangle = 1 + \langle \underline{\dim} P', \underline{\dim} P''\rangle = 1 = n$ and also $F^P_{P_1 P_0} = F^P_{P'' P'} = f_0(q)$, thus the assertion is true.

If $P' \ncong P_0$, using Proposition 2.8 we obtain that

$$F^P_{P_1 P_0} = \frac{1}{q-1}\sum_{[U]} F^{P_1}_{UP'} F^{P''}_{UP_0}\alpha_U - \frac{1}{q-1}\sum_{[U_1]} F^{P_1}_{U_1 P''} F^{P'}_{U_1 P_0}\alpha_V = S_g(P_1, P', P'', P_0) - S_g(P_1, P'', P', P_0).$$

Using Proposition 2.3, Proposition 2.2 (b) and Lemma 2.2 (b) we can see that $S_1 = S_g(P_1, P', P'', P_0) = \frac{1}{q-1}\sum_{[R]}\alpha_R$ where the sum is taken over all modules R satisfying the conditions:

(i) R is a regular (not necessarily indecomposable) module with $\underline{\dim} R = \underline{\dim} P'' - \underline{\dim} P_0$. This means that $\underline{\dim} R = \underline{\dim} P'' - \underline{\dim} P_0 = d\delta + \sigma_0 + \sigma_1 + \sigma_\infty$, where $(\underline{\dim} P'' - \underline{\dim} P_0)_{nh} = (\sigma_0, \sigma_1, \sigma_\infty)$ and $d = \dim_k \operatorname{Hom}(P_0, P'') - 1$.

(ii) The indecomposable components of R are taken from pairwise different tubes, moreover, the only non-homogeneous components are taken from the tubes $\mathcal{T}_e$ with $\sigma_e \neq 0$.

The sum is not empty if and only if $\langle \underline{\dim} P_0, \underline{\dim} P''\rangle = \dim_k \operatorname{Hom}(P_0, P'') \neq 0$. Since P_0 is a projective indecomposable corresponding to vertex 1, this means that the first component of $\underline{\dim} P''$ must be nonzero. In this case, knowing that $\underline{\dim} P'' - \underline{\dim} P_0 = d\delta + \sigma_0 + \sigma_1 + \sigma_\infty$, thus $(\underline{\dim} P'' - \underline{\dim} P_0)_{nh} = (\sigma_0, \sigma_1, \sigma_\infty)$, we will have

$$S_1 = S_g(P_1, P', P'', P_0) = {}^{w}a_d(q),$$

where $w = |\{e|\sigma_e \neq 0\}| = \operatorname{supp}(\underline{\dim} P'' - \underline{\dim} P_0)_{nh}$ and $d = \dim_k \operatorname{Hom}(P_0, P'') - 1 = \langle \underline{\dim} P_0, \underline{\dim} P''\rangle - 1$. Thus, in order to compute $S_1 = S_g(P_1, P', P'', P_0)$, we just have to know $w = \operatorname{supp}(\underline{\dim} P'' - \underline{\dim} P_0)_{nh}$ and $d = \langle \underline{\dim} P_0, \underline{\dim} P''\rangle - 1$.

An analogous statement is true for $S_2 = S_g(P_1, P'', P', P_0)$.

In case $\underline{\dim} \boldsymbol{P} < 2\boldsymbol{\delta}$, we list in Appendix B all the possible indecomposable preprojectives P together with a specific corresponding orthogonal exceptional pair (P'', P') and compute the Ringel-Hall polynomials using the above results on the sums S_g. More precisely (for S_1) we have to obtain $d = \langle \underline{\dim} P_0, \underline{\dim} P''\rangle - 1$ and thus $(\underline{\dim} P'' - \underline{\dim} P_0)_{nh}$, which provides us w (via its support). We note that the theorem is valid in this case, however the problem is that it might happen that one of the sums S_g is empty. So we have to list in Appendix B also the case $2\boldsymbol{\delta} < \underline{\dim} \boldsymbol{P} < 4\boldsymbol{\delta}$. We can see that this time all the sums S_g will be nonempty. Moreover, we can also observe that in this case (for $n = \langle \underline{\dim} P_0, \underline{\dim} P\rangle$), using the specified orthogonal exceptional pairs (P'', P'), the pair of sums

$$(S_1 = S_g(P_1, P', P'', P_0), S_2 = S_g(P_1, P'', P', P_0))$$

equals to one of the pairs

$$({}^{0}a_{\frac{n}{2}}(q), {}^{3}a_{\frac{n}{2}-2}(q)),\ ({}^{2}a_{\frac{n}{2}-1}(q), {}^{1}a_{\frac{n}{2}-1}(q)), \text{ for } n \text{ even and}$$

$$({}^{3}a_{\frac{n-1}{2}-1}(q), {}^{0}a_{\frac{n-1}{2}}(q)), ({}^{1}a_{\frac{n-1}{2}}(q), {}^{2}a_{\frac{n-1}{2}-1}(q)), \text{ for } n \text{ odd.}$$

Note by Section 1.6 that for n even

$$F^P_{P_1 P_0} = S_1 - S_2 = {}^0a_{\frac{n}{2}}(q) - {}^3a_{\frac{n}{2}-2}(q) = {}^2a_{\frac{n}{2}-1}(q) - {}^1a_{\frac{n}{2}-1}(q) = f_{n-1}(q),$$

and also for n odd

$$F^P_{P_1 P_0} = {}^3a_{\frac{n-1}{2}-1}(q) - {}^0a_{\frac{n-1}{2}}(q) = {}^1a_{\frac{n-1}{2}}(q) - {}^2a_{\frac{n-1}{2}-1}(q) = f_{n-1}(q),$$

thus our formula is valid also for $2\delta < \underline{\dim} P < 4\delta$.

In case $\underline{\mathbf{dim}} \boldsymbol{P} > \boldsymbol{4\delta}$, we have that $\underline{\dim} P = 2l\delta + \underline{\dim} \bar{P} = \underline{\dim} \bar{P}(+2l\delta)$, for a unique $\bar{P}$ with defect -2 and $2\delta < \underline{\dim} \bar{P} < 4\delta$. Thus $\bar{P}$ is in the list presented in Appendix B, and let $(\bar{P}'', \bar{P}')$ be its listed orthogonal exceptional pair. Let $\bar{n} = \langle \underline{\dim} P_0, \underline{\dim} \bar{P} \rangle$ and $\bar{d} = \langle \underline{\dim} P_0, \underline{\dim} \bar{P}'' \rangle - 1$. Then $\bar{P}_1(+2l\delta) = P_1$. By Lemma 1.12, $(P'', P') = (\bar{P}''(+l\delta), \bar{P}'(+l\delta))$ is an orthogonal exceptional pair for P, $n = \langle \underline{\dim} P_0, \underline{\dim} P \rangle = \langle \underline{\dim} P_0, 2l\delta + \underline{\dim} \bar{P} \rangle = \bar{n} + 2l$ and $d = \langle \underline{\dim} P_0, \underline{\dim} P'' \rangle - 1 = \langle \underline{\dim} P_0, l\delta + \bar{P}'' \rangle - 1 = \langle \underline{\dim} P_0, \underline{\dim} \bar{P}'' \rangle + l - 1 = \bar{d} + l$. So, we obtain

$$S_g(P_1, P', P'', P_0) = {}^wa_d(q) = {}^wa_{\bar{d}+l}(q),$$

where $w = \mathrm{supp}(\underline{\dim} P'' - \underline{\dim} P_0)_{nh}$ is the number of nonzero non-homogeneous components.

It is important to note that

$$(\underline{\dim} P'' - \underline{\dim} P_0)_{nh} = (l\delta + \underline{\dim} \bar{P}'' - \underline{\dim} P_0)_{nh} = (\underline{\dim} \bar{P}'' - \underline{\dim} P_0)_{nh},$$

thus the number of nonzero non-homogeneous components is

$$w = \mathrm{supp}(\underline{\dim} P'' - \underline{\dim} P_0)_{nh} = \mathrm{supp}(\underline{\dim} \bar{P}'' - \underline{\dim} P_0)_{nh}.$$

It follows that $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^wa_{\bar{d}}(q) = {}^wa_{d-l}(q)$. The same applies for the other sum $S_g(P_1, P'', P', P_0)$.

Since $2\delta < \underline{\dim} \bar{P} < 4\delta$, we already know that the pair of sums $(S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0), S_g(\bar{P}_1, \bar{P}'', \bar{P}', P_0))$ equals

$$({}^0a_{\frac{\bar{n}}{2}}(q), {}^3a_{\frac{\bar{n}}{2}-2}(q)) \text{ or } ({}^2a_{\frac{\bar{n}}{2}-1}(q), {}^1a_{\frac{\bar{n}}{2}-1}(q)), \text{ for } \bar{n} \text{ even}$$

and

$$({}^3a_{\frac{\bar{n}-1}{2}-1}(q), {}^0a_{\frac{\bar{n}-1}{2}}(q)) \text{ or } ({}^1a_{\frac{\bar{n}-1}{2}}(q), {}^2a_{\frac{\bar{n}-1}{2}-1}(q)), \text{ for } \bar{n} \text{ odd.}$$

So looking at the first sum, we can see the value of $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0)$ in the following cases:

- for $\bar{n}$ even $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^0a_{\frac{\bar{n}}{2}}(q)$, we have $\bar{d} = \frac{\bar{n}}{2}$;
- for $\bar{n}$ even $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^2a_{\frac{\bar{n}}{2}-1}(q)$, we have $\bar{d} = \frac{\bar{n}}{2} - 1$;
- for $\bar{n}$ odd $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^3a_{\frac{\bar{n}-1}{2}-1}(q)$, we have $\bar{d} = \frac{\bar{n}-1}{2} - 1$;
- for $\bar{n}$ odd $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^1a_{\frac{\bar{n}-1}{2}}(q)$, we have $\bar{d} = \frac{\bar{n}-1}{2}$.

But then, using the above remarks, we will have that the corresponding sums $S_g(P_1, P', P'', P_0) = {}^w a_{\bar{d}+l}(q)$ are

- for $\bar{n}$ even $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^0 a_{\frac{\bar{n}}{2}+l}(q)$;
- for $\bar{n}$ even $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^2 a_{\frac{\bar{n}}{2}-1+l}(q)$;
- for $\bar{n}$ odd $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^3 a_{\frac{\bar{n}-1}{2}-1+l}(q)$;
- for $\bar{n}$ odd $S_g(\bar{P}_1, \bar{P}', \bar{P}'', P_0) = {}^1 a_{\frac{\bar{n}-1}{2}+l}(q)$.

So the corresponding pair of sums $(S_g(P_1, P', P'', P_0), S_g(P_1, P'', P', P_0))$ equals

$$({}^0 a_{\frac{\bar{n}}{2}+l}(q), {}^3 a_{\frac{\bar{n}}{2}-2+l}(q)) \text{ or } ({}^2 a_{\frac{\bar{n}}{2}-1+l}(q), {}^1 a_{\frac{\bar{n}}{2}-1+l}(q)), \text{ for } n = \bar{n} + 2l \text{ even}$$

and

$$({}^3 a_{\frac{\bar{n}-1}{2}-1+l}(q), {}^0 a_{\frac{\bar{n}-1}{2}+l}(q)) \text{ or } ({}^1 a_{\frac{\bar{n}-1}{2}+l}(q), {}^2 a_{\frac{\bar{n}-1}{2}-1+l}(q)), \text{ for } n = \bar{n} + 2l \text{ odd}.$$

Thus we have

$$F^P_{P_1 P_0} = {}^0 a_{\frac{\bar{n}}{2}+l}(q) - {}^3 a_{\frac{\bar{n}}{2}-2+l}(q) = {}^2 a_{\frac{\bar{n}}{2}-1+l}(q) - {}^1 a_{\frac{\bar{n}}{2}-1+l}(q) = f_{\bar{n}-1+2l}(q) = f_{n-1}(q), \text{ for } n \text{ even}$$

and

$$F^P_{P_1 P_0} = {}^3 a_{\frac{\bar{n}-1}{2}-1+l}(q) - {}^0 a_{\frac{\bar{n}-1}{2}+l}(q) = {}^1 a_{\frac{\bar{n}-1}{2}+l}(q) - {}^2 a_{\frac{\bar{n}-1}{2}-1+l}(q) = f_{\bar{n}-1+2l}(q) = f_{n-1}(q), \text{ for } n \text{ odd}.$$

□

2.3.4 The polynomials $F^p_{i_0 p_1}$, where $\partial p_1 = -2$, $\partial p = -1$, $\partial i_0 = 1$

Theorem 2.7. *Suppose Q is a tame quiver and P, P_1 are indecomposable preprojectives, I_0 is indecomposable preinjective with $\partial P = -1$, $\partial P_1 = -2$, $\partial I_0 = 1$ and such that $\underline{\dim} P = \underline{\dim} P_1 + \underline{\dim} I_0$. Then $F^P_{I_0 P_1} = f_{n-1}(q)$, where $n = \langle \underline{\dim} P_1, \underline{\dim} P \rangle = \langle \underline{\dim} P, \underline{\dim} I_0 \rangle$.*

Proof. Trivially $n = \langle \underline{\dim} P_1, \underline{\dim} P \rangle = \langle \underline{\dim} P_1, \underline{\dim} P_1 + \underline{\dim} I_0 \rangle = 1 + \langle \underline{\dim} P_1, \underline{\dim} I_0 \rangle = \langle \underline{\dim} P_1 + \underline{\dim} I_0, \underline{\dim} I_0 \rangle = \langle \underline{\dim} P, \underline{\dim} I_0 \rangle$. Note that in case $n \leq 0$, thus in case $\mathrm{Hom}(P_1, P) = 0$ or $\mathrm{Hom}(P, I_0) = 0$, the Ringel-Hall number $F^P_{P_1 P_0} = 0 = f_{n-1}(q)$, so the assertion is true. From now on we will suppose that $n > 0$, thus $\mathrm{Hom}(P_1, P) \neq 0$ and $\mathrm{Hom}(P, I_0) \neq 0$.

Using a succession of reflection functors (corresponding to sources) and symmetry we can suppose without loss of generality that I_0 is the simple injective corresponding to the source 1. Note that during this process P and P_1 is never transformed into a simple projective (since we apply functors S_i^-). Also Ringel-Hall numbers, dimensions of homomorphisms and Euler products are not modified by the reflection functors.

Next, we will use the functor $S_1^- : \text{mod-}kQ\langle 1 \rangle \to \text{mod-}k\sigma_1 Q\langle 1 \rangle$. Let $P' = S_1^-(P), P_1' = S_1^-(P_1)$ and note that $\underline{\dim} P' - \underline{\dim} P_1' = \sigma_1(\underline{\dim} P - \underline{\dim} P_1) = \sigma_1(\underline{\dim} I_0) = -\underline{\dim} P_0'$, where P_0' is the simple projective in mod-$k\sigma_1 Q$ corresponding to the sink 1. Using Proposition 1.4 and the fact that submodules of preprojectives are preprojective (so $Z \in$ mod-$kQ\langle 1\rangle$) we obtain:

$$(q-1)F^P_{I_0P_1} = m^P_{P_1} = h_{P_1P} - \sum_{\underline{\dim}Z<\underline{\dim}P_1} f^{P_1}_Z \alpha_Z s^P_Z$$
$$= h_{P_1P} - \sum_{\substack{\underline{\dim}Z<\underline{\dim}P_1<\underline{\dim}P \\ Z\ preprojective}} f^{P_1}_Z \alpha_Z s^P_Z = h_{P'_1P'} - \sum_{\substack{\underline{\dim}Z'<\underline{\dim}P'<\underline{\dim}P'_1 \\ Z'\ preprojective}} f^{P'_1}_{Z'} \alpha_{Z'} s^{P'}_{Z'} = e^{P'_1}_{P'} = (q-1)F^{P'_1}_{P'P'_0}.$$

Thus using the previous subsection $F^P_{I_0P_1} = F^{P'_1}_{P'P'_0} = f_{\langle \underline{\dim}P'_1, \underline{\dim}P'\rangle - 1}(q) = f_{\langle \underline{\dim}P_1, \underline{\dim}P\rangle - 1}(q) = f_{n-1}(q)$. □

2.3.5 The polynomials $F^p_{rp_0}$, where $\partial p_0 = \partial p = -2$ and r is the symbol of a homogeneous regular

Theorem 2.8. *Suppose Q is a tame quiver and P_0, P are indecomposable preprojectives of defect -2 such that $\underline{\dim}P - \underline{\dim}P_0 = dn\delta$. Let $R_d(n) = R(n, a)$ be an indecomposable regular module of regular-length n taken from the homogeneous tube $\mathcal{T}_a$ with $\deg a = d$.*

Then $F^P_{R_d(n)P_0} = g_{d,n}(q)$, where

$$g_{d,n} = f_{dn} - f_{dn-1} + (X^d - 1)\sum_{t\geq 1} X^{d(t-1)}(f_{d(n-2t)} - f_{d(n-2t)-1}).$$

In particular $g_{d,0} = 1$. Also, for $d = n = 1$ we have $g_{1,1} = X - 3$ and in case $d = 1, n \geq 2$ the polynomial is $g_{1,n} = X^n - 3X^{n-1} + 4X^{n-2} - 4X^{n-3} + \cdots + (-1)^{n-1}4X + (-1)^n 4$.

Proof. The condition $\underline{\dim}P - \underline{\dim}P_0 = dn$ implies that $\langle \underline{\dim}P_0, \underline{\dim}P\rangle = \langle \underline{\dim}P_0, \underline{\dim}P_0 + dn\delta\rangle = 1 + 2dn > 0$, thus $\mathrm{Hom}(P_0, P) \neq 0$.

Using a succession of reflection functors (corresponding to sinks) and symmetry we can suppose without loss of generality that P_0 is simple projective corresponding to a sink. Note that during this process P and $R_d(n)$ is never transformed into a simple projective. So Ringel-Hall numbers, dimensions of homomorphisms and Euler products are not modified by the reflection functors.

Let (P'', P') be an arbitrary preprojective orthogonal exceptional pair corresponding to P. The existence of such a pair is guaranteed by the results in [60], the dimension of P being bigger then δ.

Note that in this case $\underline{\dim}P' - \underline{\dim}P_0$ (and $\underline{\dim}P'' - \underline{\dim}P_0$) is a positive root of defect 1, since

$$\langle \underline{\dim}P' - \underline{\dim}P_0, \underline{\dim}P' - \underline{\dim}P_0\rangle = \langle dn\delta - \underline{\dim}P'', dn\delta - \underline{\dim}P''\rangle = 1.$$

Denote by I'_0 (respectively by I''_0) the preinjective indecomposable of defect 1 having dimension $\underline{\dim}P'' - \underline{\dim}P_0 = dn\delta - \underline{\dim}P'$ (respectively $\underline{\dim}P' - \underline{\dim}P_0$).

We apply Proposition 2.8 for P, P_0 and $X = R$ obtaining

$$F^P_{R_d(n)P_0} = \frac{1}{\alpha_{R_d(n)}}\sum_{[U]} F^{R_d(n)}_{UP'} F^{P''}_{UP_0}\alpha_U - \frac{1}{\alpha_{R_d(n)}}\sum_{[V]} F^{R_d(n)}_{VP''} F^{P'}_{VP_0}\alpha_V$$
$$= S_g(R_d(n), P', P'', P_0) - S_g(R_d(n), P'', P', P_0).$$

Looking at the first sum, if the uniserial $R_d(n)$ projects on U (with $\partial U = 1$), then up to isomorphism U must take the form $R_d(t)\oplus I'_t$ where $R_d(t) = R(t,a)$ and I'_t is an indecomposable preinjective of defect 1 and $\underline{\dim} I'_t = d(n-t)\delta - \underline{\dim} P' = \underline{\dim} P'' - dt\delta - \underline{\dim} P_0$, so $I'_t = I'_0(-dt\delta)$ and $d(n-t)\delta > \underline{\dim} P'$. We deduce using Proposition 2.3 and associativity of the Ringel-Hall product that

$$F^{R_d(n)}_{R_d(t)\oplus I'_t P'} = F^{R_d(n)}_{R_d(t)R_d(n-t)} F^{R_d(n-t)}_{I'_t P'}$$
$$= 1\cdot\frac{1}{q-1}\alpha_{R_d(n-t)} = \frac{1}{q-1}(q^{d(n-t)} - q^{d(n-t-1)}) = \frac{1}{q-1}q^{d(n-t-1)}(q^d-1)$$

and

$$F^{P''}_{R_d(t)\oplus I'_t P_0} = F^{P''}_{R_d(t)P''(-dt\delta)} F^{P''(-dt\delta)}_{I'_t P_0} = F^{P''(-dt\delta)}_{I'_t P_0}.$$

Also note that $\alpha_{R_d(t)\oplus I'_t} = q^{\langle \underline{\dim} R_d(t), \underline{\dim} I'_t\rangle}\alpha_{R_d(t)}\alpha_{I'_t} = q^{dt}(q^{dt}-q^{d(t-1)})(q-1) = q^{d(2t-1)}(q^d-1)(q-1)$ for $t>0$, $\alpha_{R_d(n)} = q^{dn} - q^{d(n-1)}$ and $\alpha_{I''_0} = q-1$ (see Lemma 1.4). This means that

$$S_g(R_d(n), P', P'', P_0) = F^{P''}_{I'_0 P_0} + (q^d-1)\sum_{t\geq 1} q^{d(t-1)} F^{P''(-dt\delta)}_{I'_0(-dt\delta)P_0}$$

and

$$S_g(R_d(n), P'', P', P_0) = F^{P'}_{I''_0 P_0} + (q^d-1)\sum_{t\geq 1} q^{d(t-1)} F^{P'(-dt\delta)}_{I''_0(-dt\delta)P_0}.$$

But $\langle \underline{\dim} P_0, \underline{\dim} P''\rangle = \langle \underline{\dim} P' + \underline{\dim} P'' - nd\delta, \underline{\dim} P''\rangle = 1 + nd$ (since $\langle \underline{\dim} P', \underline{\dim} P''\rangle = 0$) and in the same way $\langle \underline{\dim} P_0, \underline{\dim} P'\rangle = nd$ (since $\langle \underline{\dim} P'', \underline{\dim} P'\rangle = -1$). It follows that $F^{P''(-dt\delta)}_{I'_t P_0} = f_{\langle P_0, P''(-dt\delta)\rangle - 1}(q) = f_{d(n-2t)}(q)$ and $F^{P'(-dt\delta)}_{I''_t P_0} = f_{d(n-2t)-1}(q)$, so

$$F^P_{R_d(n)P_0} = f_{dn}(q) - f_{dn-1}(q) + (q^d-1)\sum_{t\geq 1} q^{d(t-1)}(f_{d(n-2t)}(q) - f_{d(n-2t)-1}(q)).$$

In particular, when $d=1$ we have $F^P_{RP_0} = q^n - 3q^{n-1} + 4q^{n-2} - 4q^{n-3} + \cdots + (-1)^{n-1}4q + (-1)^n 4$. □

2.3.6 The polynomials $F^r_{i_0 p}$, where $\partial p = -2$, $\partial i_0 = 2$ and r is the symbol of a homogeneous regular

Theorem 2.9. *Suppose Q is a tame quiver, I_0 an indecomposable preinjective of defect 2 and P an indecomposable preprojective of defect -2 such that $\underline{\dim} P + \underline{\dim} I_0 = dn\delta$. Let $R_d(n) = R(n,a)$ be an indecomposable regular module of regular-length n taken from the homogeneous tube $\mathcal{T}_a$ with $\deg a = d$.*

Then

$$F^{R_d(n)}_{I_0 P} = \frac{q^{dn} - q^{d(n-1)}}{q-1} g_{d,n}(q).$$

Proof. Using a succession of reflection functors (corresponding to sources), renumbering of the vertices and symmetry we can suppose without loss of generality that I_0 is the simple injective corresponding to the source 1. Note that during this process P and $R_d(n)$ is never transformed into a simple projective (since we apply functors S_i^-). Also Ringel-Hall numbers, dimensions of homomorphisms and Euler products are not modified by the reflection functors.

Next, we will use the functor $S_1^- : \text{mod-}kQ\langle 1\rangle \to \text{mod-}k\sigma_1 Q\langle 1\rangle$. Let $P' = S_1^-(P)$, $R'_d(n) = S_1^-(R_d(n))$ (since homogeneity, degree of the tube and regular-length remain invariant) and note that $\underline{\dim} R'_d(n) - \underline{\dim} P' = \sigma_1(\underline{\dim} R_d(n) - \underline{\dim} P) = \sigma_1(\underline{\dim} I_0) = -\underline{\dim} P'_0$, where P'_0 is the simple projective in mod-$k\sigma_1 Q$ corresponding to the sink 1. Using Proposition 1.4 and the fact that submodules of regulars have no preinjective component (so $Z \in \text{mod-}kQ\langle 1\rangle$) we obtain:

$$
\begin{aligned}
(q-1)F_{I_0 P}^{R_d(n)} = m_P^{R_d(n)} &= h_{PR_d(n)} - \sum_{\underline{\dim} Z < \underline{\dim} P} f_Z^P \alpha_Z s_Z^{R_d(n)} \\
&= h_{PR_d(n)} - \sum_{\substack{\underline{\dim} Z < \underline{\dim} P < \underline{\dim} R_d(n) \\ Z \text{ without preinjective components}}} f_Z^P \alpha_Z s_Z^{R_d(n)} \\
&= h_{P' R'_d(n)} - \sum_{\substack{\underline{\dim} Z' < \underline{\dim} R'_d(n) < \underline{\dim} P' \\ Z' \text{ without preinjective components}}} f_{Z'}^{P'} \alpha_{Z'} s_{Z'}^{R'_d(n)} = e_{R'_d(n)}^{P'} = \alpha_{R'_d(n)} F_{R'_d(n) P'_0}^{P'} = (q^{dn} - q^{d(n-1)}) g_{d,n}(q).
\end{aligned}
$$

□

2.3.7 The polynomials $F_{rp_0}^p$, where $\partial p_0 = \partial p = -2$ and r is the symbol of a non-homogeneous regular

Theorem 2.10. *Suppose Q is a tame quiver and P_0, P are indecomposable preprojectives of defect -2 such that $\underline{\dim} P - \underline{\dim} P_0 = \underline{\dim} R$, where R is an indecomposable regular module taken from a non-homogeneous tube such that P projects to the regular-top of R, thus $\langle \underline{\dim} P, \underline{\dim}\,\text{top}_r R\rangle \neq 0$. Then $F_{RP_0}^P = h_n(q)$, where $n = \lfloor \frac{1}{2}\langle \underline{\dim} P_0, \underline{\dim} P\rangle \rfloor$ and $h_0 = 1$, $h_1 = X - 2$ and for $n \geq 2$*

$$h_n = X^n - 4(X^{n-1} - 2X^{n-2} + 3X^{l-3} - \cdots + (-1)^{n-2}(n-1)X) + (-1)^n 2n = f_n - f_{n-1} + (-1)^{n-1}.$$

Proof. Suppose Q is a tame quiver and P_0, P are indecomposable preprojectives of defect -2 and R is a non-homogeneous regular indecomposable taken from a tube e such that $\underline{\dim} P - \underline{\dim} P_0 = \underline{\dim} R$.

Using a succession of reflection functors (corresponding to sinks) and symmetry, we can modify the orientation of Q such that it possesses a unique sink and P_0 is exactly the simple projective corresponding to this sink. Note that during this process P and R is never transformed into a simple projective. Indeed, the relation $\underline{\dim} P = \underline{\dim} P_0 + \underline{\dim} R$ is kept even after reflections, so P can't be simple. This means that Ringel-Hall numbers, dimensions of homomorphisms and Euler products are not modified by the reflection functors.

We will use again a specific preprojective orthogonal exceptional pair (P'', P') corresponding to P. Such a pair will almost always exists (for the given orientation and context) and will be explicitly given in Section C.1. Only in a few cases for some very small dimensions might happen that there is no such pair. For these cases the corresponding Ringel-Hall polynomial will be obtained by direct calculation.

We apply Proposition 2.8 for P, P_0 and $X = R$ obtaining

$$F_{RP_0}^P = \frac{1}{\alpha_R} \sum_{[U]} F_{UP'}^R F_{UP_0}^{P''} \alpha_U - \frac{1}{\alpha_R} \sum_{[V]} F_{VP''}^R F_{VP_0}^{P'} \alpha_V$$

$$= S_g(R, P', P'', P_0) - S_g(R, P'', P', P_0).$$

If the uniserial R projects on U (with $\partial U = 1$), then up to isomorphism U is either of the form I' or of the form $R' \oplus I_{R'}$, where R' is an indecomposable regular factor of R (it can be also 0), with the kernel R'' also indecomposable (due to uniseriality) and where I', $I_{R'}$ are indecomposable preinjectives of defect 1 such that $\underline{\dim} I_{R'} + \underline{\dim} R' = \underline{\dim} I' = \underline{\dim} R - \underline{\dim} P'$.

So, using Proposition 2.1, Corollary 2.1, Corollary 2.2 and the previous results we get:

$$\begin{aligned}
S_1 = S_g(R, P', P'', P_0) &= \frac{1}{\alpha_R}\Big(F^R_{I'P'}F^{P''}_{I'P_0}\alpha_{I'} + \sum_{[R']} F^R_{R'\oplus I_{R'}P'}F^{P''}_{R'\oplus I_{R'}P_0}\alpha_{R'\oplus I_{R'}}\Big) \\
&= \frac{1}{\alpha_R}\Big(F^R_{I'P'}F^{P''}_{I'P_0}\alpha_{I'} + \sum_{[R']} F^R_{R'R''}F^{R''}_{I_{R'}P'}F^{P''}_{R'P_1}F^{P_1}_{I_{R'}P_0}q^{\langle \underline{\dim} R', \underline{\dim} I_{R'}\rangle}\alpha_{R'}\alpha_{I_{R'}}\Big) \\
&= \delta_{\mathrm{top}R, R_{P'}(1,e)} f_{\langle \underline{\dim} P_0, \underline{\dim} P''\rangle - 1}(q) + \\
&+ \frac{1}{\alpha_R} \sum_{\substack{[R'],[R'']\in\mathcal{T}_e \text{ indec.} \\ 0\to R''\to R\to R'\to 0 \text{ exact} \\ \underline{\dim} R'' > \underline{\dim} P' \\ \mathrm{top}R'' = R_{P'}(1,e) \\ \mathrm{top}R' = R_{P''}(1,e)}} \alpha_{R''}\alpha_{R'} f_{\langle \underline{\dim} P_0, \underline{\dim} P'' - \underline{\dim} R'\rangle - 1}(q) q^{\langle \underline{\dim} R', \underline{\dim} P'' - \underline{\dim} R' - \underline{\dim} P_0\rangle}. \qquad (*)
\end{aligned}$$

Similarly,

$$\begin{aligned}
S_2 = S_g(R, P'', P', P_0) &= \delta_{\mathrm{top}R, R_{P''}(1,e)} f_{\langle \underline{\dim} P_0, \underline{\dim} P'\rangle - 1}(q) + \\
&+ \frac{1}{\alpha_R} \sum_{\substack{[R'],[R'']\in\mathcal{T}_e \text{ indec.} \\ 0\to R''\to R\to R'\to 0 \text{ exact} \\ \underline{\dim} R'' > \underline{\dim} P'' \\ \mathrm{top}R'' = R_{P''}(1,e) \\ \mathrm{top}R' = R_{P'}(1,e)}} \alpha_{R''}\alpha_{R'} f_{\langle \underline{\dim} P_0, \underline{\dim} P' - \underline{\dim} R'\rangle - 1}(q) q^{\langle \underline{\dim} R', \underline{\dim} P' - \underline{\dim} R' - \underline{\dim} P_0\rangle}. \qquad (**)
\end{aligned}$$

In order to perform the explicit calculations based on the formulas above we need to look separately to each tame case, since as presented in the preliminaries, the system of ranks of the regular non-homogeneous tubes is different in the various tame cases.

In each tame case we will follow the steps below:

1. Consider the set $\mathcal{P}$ of all isomorphism classes of preprojective indecomposable modules P such that:

   (a) $\partial P = -2$,

   (b) $\underline{\dim} P - \underline{\dim} P_0 = \underline{\dim} R$ is the dimension of a regular non-homogeneous indecomposable R, thus it takes the unique form $2d_P\delta + \sigma_P$, where $0 \le \sigma_P < 2\delta$ if nonzero is either a positive real root or δ; note that P projects to the regular top of R in case σ_P is a positive real root, however if $\sigma_P = 0$ (or δ) then the condition requiring P to project onto the regular top of R is satisfied if and only if the module R taken from a non-homogeneous tube $\mathcal{T}_e$ of rank m has regular length $2d_P m$ (or $(2d_P + 1)m$) and regular top $R^1_P(1,e)$ or $R^2_P(1,e)$ (see Lemma 2.2).

Let $\Sigma_Q = \{\sigma_P | [P] \in \mathcal{P}\}$, so it is the set of "remainders" modulo 2δ of $\underline{\dim} P - \underline{\dim} P_0$. It is important to note that Σ_Q will be a finite set.

For each $\sigma \in \Sigma_Q$ denote by P_σ the preprojective indecomposable with $\underline{\dim} P_\sigma = \underline{\dim} P_0 + \sigma$. Define $\Sigma'_Q = \{\sigma \in \Sigma_Q | P_\sigma$ has an orthogonal exceptional pair$\}$ and $\Sigma''_Q = \Sigma_Q \setminus \Sigma'_Q$.

For $\sigma \in \Sigma''_Q$, thus in cases P_σ does not have an orthogonal exceptional pair (this might happen only in a few cases for some very small dimensions), the corresponding Ringel-Hall polynomial will be obtained by direct calculation.

Define for $\sigma \in \Sigma'_Q$

$$\mathcal{P}_\sigma = \{[P] \in \mathcal{P} | \sigma_P = \sigma\} = \{[P_\sigma(+2t\delta)] | t \in \mathbb{N}\}$$

and for $\sigma \in \Sigma''_Q$

$$\mathcal{P}_{\sigma+2\delta} = \{[P_\sigma(+2t\delta)] | t \in \mathbb{N}^*\}.$$

This separation of cases is done since in case $\sigma \in \Sigma''_Q$, although P_σ does not have an orthogonal exceptional pair, we will see that $P_\sigma(+2\delta)$ possesses such a pair.

2. For $\sigma \in \Sigma'_Q$ and $[P_\sigma(+2t\delta)] \in \mathcal{P}_\sigma$ the considered generic orthogonal exceptional pair will be $(P''_\sigma(+t\delta), P'_\sigma(+t\delta))$, where (P''_σ, P'_σ) is a certain orthogonal exceptional pair of P_σ given in Section C.1.

   For $\sigma \in \Sigma''_Q$ and $[P_\sigma(+2t\delta)] \in \mathcal{P}_{\sigma+2\delta}$ (where $t \geq 1$) the considered generic orthogonal exceptional pair will be $(P''_{\sigma+2\delta}(+(t-1)\delta), P'_{\sigma+2\delta}(+(t-1)\delta))$, where $(P''_{\sigma+2\delta}, P'_{\sigma+2\delta})$ is a certain orthogonal exceptional pair of $P_\sigma(+2\delta)$.

3. For elements in a class $\mathcal{P}_\sigma$ (or $\mathcal{P}_{\sigma+2\delta}$) using the generic orthogonal exceptional pair and the formulas ($*$), ($**$) we can compute the Ringel-Hall polynomial by $F^P_{RP_0} = S_g(R, P', P'', P_0) - S_g(R, P'', P', P_0) = S_1 - S_2$.

We present in Section C.1 the detailed calculation based on the steps above in the case $\widetilde{\mathbb{E}}_6$. The other tame cases follow an analogous, however much more lengthier path, so we will omit the details and just list the final results in Section C.2.

The calculations above and their results confirm us that the assertion of the theorem is true. □

2.3.8 The polynomials $F^r_{i_0 p}$, where $\partial p = -2$, $\partial i_0 = 2$ and r is the symbol of a non-homogeneous regular

Theorem 2.11. *Suppose Q is a tame quiver, I_0 an indecomposable preinjective of defect 2 and P an indecomposable preprojective of defect -2 such that $\underline{\dim} P + \underline{\dim} I_0 = \underline{\dim} R$, where R is an indecomposable regular module taken from a non-homogeneous tube such that $\langle \underline{\dim} P, \underline{\dim}\ \mathrm{top}_r R\rangle \neq 0$. Then $F^R_{I_0 P} = q^{m-1} h_n(q)$, where $(m-1)\delta < \underline{\dim} R \leq m\delta$ and $n = \lfloor -\frac{1}{2}\langle \underline{\dim} I_0, \underline{\dim} P\rangle \rfloor$.*

Proof. Using a succession of reflection functors (corresponding to sources), renumbering of the vertices and symmetry, we can suppose without loss of generality that I_0 is the simple injective corresponding to the source 1. Note that during this process P and R is never transformed into a simple projective (since we apply functors S_i^-). Also Ringel-Hall numbers, dimensions of homomorphisms and Euler products are not modified by the reflection functors.

Next, we will use the functor $S_1^- : \text{mod-}kQ\langle 1\rangle \to \text{mod-}k\sigma_1 Q\langle 1\rangle$. Let $P' = S_1^-(P), R' = S_1^-(R)$ (R' being non-homogeneous regular indecomposable) and note that $\underline{\dim}R' - \underline{\dim}P' = \sigma_1(\underline{\dim}R - \underline{\dim}P) = \sigma_1(\underline{\dim}I_0) = -\underline{\dim}P'_0$, where P'_0 is the simple projective in mod-$k\sigma_1 Q$ corresponding to the sink 1. Using Proposition 1.4 and the fact that submodules of regulars have no preinjective component (so $Z \in \text{mod-}kQ\langle 1\rangle$) we obtain:

$$\begin{aligned}(q-1)F^R_{I_0 P} = m^R_P &= h_{PR} - \sum_{\underline{\dim}Z < \underline{\dim}P} f^P_Z \alpha_Z s^R_Z \\ &= h_{PR} - \sum_{\substack{\underline{\dim}Z < \underline{\dim}P < \underline{\dim}R \\ Z \text{ without preinjective components}}} f^P_Z \alpha_Z s^R_Z \\ &= h_{P'R'} - \sum_{\substack{\underline{\dim}Z' < \underline{\dim}R' < \underline{\dim}P'_1 \\ Z' \text{ without preinjective components}}} f^{P'}_{Z'} \alpha_{Z'} s^{R'}_{Z'} = e^{P'}_{R'} = \alpha_{R'} F^{P'}_{R'P'_0} = q^{m-1}(q-1)h_n(q),\end{aligned}$$

since R' has the same regular-length as R and also

$$\langle \underline{\dim}I_0, \underline{\dim}P\rangle = \langle \sigma_1(\underline{\dim}I_0), \sigma_1(\underline{\dim}P)\rangle = \langle -\underline{\dim}P'_0, \underline{\dim}P'\rangle.$$

□

Remark 2.1. (a) We have noticed that for the discrete type Ringel-Hall polynomials we have $f_n(1) = h_n(1) = (-1)^n$. This happens also in the Dynkin case (observed by Ringel in [42]). However, for the continuous type Ringle-Hall polynomial family $g_{d,n}(1) = 2(-1)^{dn}$ and $g_{d,0}(1) = 1$.

(b) We verified via a computer program that the polynomials f_n are irreducible (as integer polynomials) up to a very high degree. However we do not have a proof in general for the irreducibility.

Chapter 3

Ringel-Hall polynomials in GR theory

Ringel-Hall polynomials appear in various contexts: as mentioned before, they are the structure constants of quantum groups, they are used in the theory of cluster algebras and they can also be used successfully to investigate the structure of the module category. In [45] Ringel used Ringel-Hall polynomials over Dynkin quivers (thus of finite representation type) to reprove Bo Chen's theorem, which states that in the Dynkin case the GR factor and the GR submodule form an orthogonal exceptional pair. Extending Ringel's idea to tame quivers, the results of this chapter published in [57] describe properties of the Gabriel-Roiter submodules of some specific modules over Euclidean quivers, using tame Ringel-Hall polynomials in the proofs.

We will refer to Section 1.11 for all the basic definitions of GR theory.

Let Q be a simply-laced acyclic tame quiver, k a field and consider the module category mod-kQ. Let x be a positive real root with negative defect. Throughout this chapter we will denote by $P^k(x)$ the (up to isomorphism) unique indecomposable preprojective of dimension x.

In the first section (Theorem 3.1 and Theorem 3.2) we prove that the GR inclusions in preprojective indecomposables and homogeneous modules of dimension δ as well as their GR measures are field independent. A similar result for Dynkin quivers was obtained by Ringel in [45]. More precisely, our first theorem asserts that the GR measure of $P^k(y)$ is independent from k. Moreover, if $P^k(x) \hookrightarrow P^k(y)$ is a GR inclusion, the roots a depend also only on the root b and not on k. The second theorem claims that GR measures of non-isomorphic homogeneous modules of dimension δ are equal and field independent. For a GR inclusion $P^k(x) \hookrightarrow R$, where R is homogeneous of dimension δ, the root a is independent from k and also from the isomorphism class of R. In order to prove these theorems we use Ringel's ideas from [45], Bo Chen's results from [13], Ringel-Hall polynomials from the previous chapter and results from algebraic geometry.

As an application of the theorems above we will prove in the second section a result by Bo Chen in [15] in a more general context: our result is valid also for the case $\widetilde{\mathbb{E}}_8$ (this case is missing from [15]) and it is field independent (in [15] k is algebraically closed). More precisely we prove in Theorem 3.4 that a GR submodule P of a homogeneous module R of dimension δ has defect -1. As a consequence we obtain a Kronecker pair $(R/P, P)$ and in this way we can embed the module category of the Kronecker algebra into mod-kQ, by sending the simple projective to P and the simple injective to R/P (see Section 1.10). The proof of the theorem above follows the idea of Ringel from [45]: one compares all possible Ringel-Hall polynomials listed in Section 2.2 with the special form they take in case of a GR inclusion.

3.1 GR measure of preprojectives and homogeneous modules of dimension δ

The first aim of this section is to prove that the GR inclusions in preprojectives are field independent. In particular the GR measure of preprojectives is also field independent. A similar statement is true for all the indecomposables in the Dynkin case (see [45]). Our approach for the tame case uses Ringel's ideas from [45] but also introduces some new concepts. Note that contrary to the Dynkin case, we cannot use Schofield short exact sequences.

First, we consider some conventions. Consider a representation M of dimension $x = (x_i)_{i \in Q_0}$ such that its linear application $k^{x_{t(\alpha)}} \to k^{x_{h(\alpha)}}$ corresponding to the arrow α is given by the matrix A_α (in the canonical basis). An endomorphism of this representation (using the canonical basis) can be identified with a collection $(X_i)_{i \in Q_0}$ of square matrices X_i of dimension x_i which satisfy the relations $A_\alpha X_{t(\alpha)} = X_{h(\alpha)} A_\alpha$. These relations induce a homogeneous linear system of equations with the unknowns being the elements of X_i. Denote by A_M the matrix of this system. Trivially we have $\dim_k \operatorname{End}(M) = \operatorname{corank} A_M$.

Let k be a field with prime field k_0, x a positive real root such that $\partial x < 0$ and $P^k(x)$ the preprojective indecomposable representation over k with dimension x (unique up to isomorphism). In case we use the representation $P^{k_0}(x)$ over the prime field k_0 of k, we agree that the representation $P^k(x)$ is constructed in the following way: for $\alpha \in Q_1$ consider the linear application $k^{x_{t(\alpha)}} \to k^{x_{h(\alpha)}}$ having the same matrix in the canonical basis as the linear application $k_0^{x_{t(\alpha)}} \to k_0^{x_{h(\alpha)}}$ in the representation $P^{k_0}(x)$. Note that $P^k(x)$ is an indecomposable representation over k of dimension x (since $A_{P^k(x)} = A_{P^{k_0}(x)}$, so $\dim_k \operatorname{End}(P^k(x)) = \operatorname{corank} A_{P^k(x)} = \operatorname{corank} A_{P^{k_0}(x)} = 1 = \dim_k \operatorname{End}(P^{k_0}(x))$). A second convention is that if we use the rational representation $P^{\mathbb{Q}}(x)$ with rational matrices corresponding to the arrows (in the canonical base) A_α, then for a big enough prime p the representation $P^{\mathbb{F}_p}(x)$ has matrices $A_\alpha \bmod p$. Note also that this representation $P^{\mathbb{F}_p}(x)$ is indecomposable for p big enough.

The conventions above and the fact that a rational matrix taken modulo a big enough prime keeps its rank imply the lemma below.

Lemma 3.1. *(a) If there is a monomorphism $P^{k_0}(x) \to P^{k_0}(y)$ then there is a monomorphism $P^k(x) \to P^k(y)$.*

(b) If there is a monomorphism $P^{\mathbb{Q}}(x) \to P^{\mathbb{Q}}(y)$ then there is a monomorphism $P^{\mathbb{F}_p}(x) \to P^{\mathbb{F}_p}(y)$ for a prime p big enough.

The next lemma is a straightforward generalization of the corresponding result by Ringel in the Dynkin case (see [45]).

Lemma 3.2. *If there is a GR inclusion $P^k(x) \to P^k(y)$ then there is a monomorphism $P^{k_0}(x) \to P^{k_0}(y)$.*

Using the existence of Ringel-Hall polynomials in the tame case we obtain:

Lemma 3.3. *If there is a monomorphism $P^k(x) \to P^k(y)$ for k finite then there is a monomorphism $P^{k'}(x) \to P^{k'}(y)$ for k' finite and $|k'|$ big enough.*

Proof. Denote by α the decomposition symbol of the cokernel in the monomorphism $P^k(x) \to P^k(y)$. Using Theorem 1.1 there is a rational polynomial $F^y_{\alpha x}$ such that for any field k' with $q' \geq q$ elements $F^y_{\alpha x}(q') = \sum_{A \in S(\alpha,k')} F^{P^{k'}(y)}_{AP^{k'}(x)}$. Due to our condition, $F^y_{\alpha x}$ is a nonzero polynomial, so for q' big enough $F^y_{\alpha x}(q')$ is also nonzero. This implies our statement. □

Using all the lemmas above we obtain the proposition below.

Proposition 3.1. *(a) Consider a field k and its prime field k_0. Then $\mu(P^{k_0}(x)) = \mu(P^k(x))$, moreover we have a GR inclusion $P^{k_0}(x) \to P^{k_0}(y)$ if and only if we have a GR inclusion $P^k(x) \to P^k(y)$.*

(b) Consider two fields k, k' with prime characteristic and a third field k'' of characteristic 0. Then $\mu(P^k(x)) = \mu(P^{k'}(x)) \geq \mu(P^{k''}(x)) = \mu(P^{\mathbb{Q}}(x))$, moreover we have a GR inclusion $P^k(x) \to P^k(y)$ if and only if we have a GR inclusion $P^{k'}(x) \to P^{k'}(y)$.

Proof. (a) If $\mu(P^k(x)) = \{n_1, \ldots, n_t\}$ then (using the fact that submodules of preprojectives are preprojective) there is a sequence of GR inclusions $P^k(x_1) \to \ldots \to P^k(x_t) = P^k(x)$ with $|P^k(x_i)| = n_i$. By Lemma 3.2 there is a chain of monomorphisms $P^{k_0}(x_1) \to \ldots \to P^{k_0}(x_t) = P^{k_0}(x)$ with $|P^{k_0}(x_i)| = n_i$. It follows that $\mu(P^k(x)) \leq \mu(P^{k_0}(x))$. In the same manner, using Lemma 3.1 (a) we obtain that $\mu(P^{k_0}(x)) \leq \mu(P^k(x))$, so $\mu(P^k(x)) = \mu(P^{k_0}(x))$.

Suppose now that we have a GR inclusion $P^k(x) \to P^k(y)$, so $\mu(P^k(y)) = \mu(P^k(x)) \cup \{|y|\}$. Then by Lemma 3.2 we have a monomorphism $P^{k_0}(x) \to P^{k_0}(y)$. Moreover, using the first part of our statement we have $\mu(P^{k_0}(y)) = \mu(P^k(y)) = \mu(P^k(x)) \cup \{|y|\} = \mu(P^{k_0}(x)) \cup \{|y|\}$, which means that we have a GR inclusion $P^{k_0}(x) \to P^{k_0}(y)$. Conversely we proceed in the same way.

(b) Using (a) we have $\mu(P^{\mathbb{Q}}(x)) = \mu(P^{k''}(x))$. Suppose that char $k = p$ and char $k' = q$. Then again by (a) we have $\mu(P^k(x)) = \mu(P^{\mathbb{F}_p}(x))$ and $\mu(P^{k'}(x)) = \mu(P^{\mathbb{F}_q}(x))$. Using Lemma 3.3 one gets that $\mu(P^{\mathbb{F}_p}(x)) \leq \mu(P^{\mathbb{F}_{q^l}}(x))$ for l big enough. But as before $\mu(P^{\mathbb{F}_{q^l}}(x)) = \mu(P^{\mathbb{F}_q}(x))$, so $\mu(P^{\mathbb{F}_p}(x)) \leq \mu(P^{\mathbb{F}_q}(x))$, which implies (changing p with q) that $\mu(P^{\mathbb{F}_p}(x)) = \mu(P^{\mathbb{F}_q}(x))$. By Lemma 3.1 (b) we get that $\mu(P^{\mathbb{Q}}(x)) \leq \mu(P^{\mathbb{F}_r}(x))$ for a big enough prime r.

Suppose now that we have a GR inclusion $P^k(x) \to P^k(y)$, so $\mu(P^k(y)) = \mu(P^k(x)) \cup \{|y|\}$. Then by Lemma 3.2 we have a monomorphism $P^{\mathbb{F}_p}(x) \to P^{\mathbb{F}_p}(y)$, so by Lemma 3.3 we also have a monomorphism $P^{\mathbb{F}_{q^l}}(x) \to P^{\mathbb{F}_{q^l}}(y)$ for l big enough. Moreover, using the first part of our statement we have $\mu(P^{\mathbb{F}_{q^l}}(y)) = \mu(P^k(y)) = \mu(P^k(x)) \cup \{|y|\} = \mu(P^{\mathbb{F}_{q^l}}(x)) \cup \{|y|\}$, which means that we have a GR inclusion $P^{\mathbb{F}_{q^l}}(x) \to P^{\mathbb{F}_{q^l}}(y)$. By Lemma 3.2 and Lemma 3.1 (a) this implies a monomorphism $P^{k'}(x) \to P^{k'}(y)$ which is in fact a GR inclusion, since $\mu(P^{k'}(y)) = \mu(P^k(y)) = \mu(P^k(x)) \cup \{|y|\} = \mu(P^{k'}(x)) \cup \{|y|\}$.

□

The proposition above and Proposition 1.9 together imply:

Proposition 3.2. *Let k' be a finite field with q' elements. If we have a GR inclusion $P^{k'}(x) \to P^{k'}(y)$ then there is a prime power q_0 such that for every finite field k with $q \geq q_0$ elements it is true that*

$$u^{P^k(y)}_{P^k(x)} = \frac{q^h - q^s}{q-1} = q^s(q^{h-s-1} + \ldots + q + 1),$$

where $h = \dim_k \operatorname{Hom}(P^k(x), P^k(y)) > s = \dim_k \operatorname{Sing}(P^k(x), P^k(y))$ are field independent.

Proof. By Theorem 1.1 there is a rational polynomial $f = \sum_\alpha F^y_{\alpha x}$, where α runs over all decomposition symbols of dimension $y-x$. We also know that there is a prime power q_0 such that $n_\alpha(q) = |S(\alpha,k)| \neq 0$ for all decomposition symbols α of dimension $y-x$ and every finite field k with $q \geq q_0$ elements.

Since we have a GR inclusion $P^{k'}(x) \to P^{k'}(y)$ then using the proposition above we have a GR inclusion $P^k(x) \to P^k(y)$ for every finite field k and by Proposition 1.9 (e), (f)

$$u^{P^k(y)}_{P^k(x)} = q^{s_k}(q^{h-s_k-1} + \ldots + q + 1),$$

where $h = \dim_k \operatorname{Hom}(P^k(x), P^k(y)) = \langle x, y\rangle$ is field independent (preprojectives being directing) and $s_k = \dim_k \operatorname{Sing}(P^k(x), P^k(y))$. Obviously we also have that $u^{P^k(y)}_{P^k(x)} = f(q)$ for a finite field k with $q \geq q_0$ elements. Since $s_k < h$, there is a value $s < h$ such that

$$u^{P^k(y)}_{P^k(x)} = q^s(q^{h-s-1} + \ldots + q + 1)$$

for infinitely many values q, but this means that $f = \frac{X^h - X^s}{X-1}$. □

Our next proposition uses the geometrical Lemma 1.15.

Proposition 3.3. *If there is a GR inclusion $P^k(x) \to P^k(y)$ for a finite field k, then there is a monomorphism $P^{\mathbb{C}}(x) \to P^{\mathbb{C}}(y)$.*

Proof. Due to Ringel (see [36]) we know that $P^{\mathbb{C}}(x)$ and $P^{\mathbb{C}}(y)$ are tree modules, so we can suppose that the matrices corresponding to the arrows contain only 0 and 1. These representations exist also over $\mathbb{Q}$ and modulo p (with the same matrices) in case p is a big enough prime. We fix such a p from now on.

Define $X = \{N \in \text{mod-}\mathbb{Q}Q \mid N \leq P^{\mathbb{Q}}(y), N \cong P^{\mathbb{Q}}(x)\}$. Since we have that $X = \{N \in \text{mod-}\mathbb{Q}Q \mid N \leq P^{\mathbb{Q}}(y),\ \underline{\dim} N = x,\ \dim_k \operatorname{End}(N) = 1\}$, one can see that X is a (locally closed) subvariety of the quiver Grassmannian $Gr_x(P^{\mathbb{Q}}(y)) = \{N \in \text{mod-}\mathbb{Q}Q \mid N \leq P^{\mathbb{Q}}(y),\ \underline{dim} N = x\}$. One can also see that X has in fact a $\mathbb{Z}$-form (see [47]) and in this way $X(\mathbb{F}_{p^l}) = \{N \in \text{mod-}\mathbb{F}_{p^l}Q \mid N \leq P^{\mathbb{F}_{p^l}}(y),\ N \cong P^{\mathbb{F}_{p^l}}(x)\}$.

Since we have a GR inclusion $P^k(x) \to P^k(y)$ for a finite field k, then using Proposition 3.2 we obtain for p big enough that

$$|X(\mathbb{F}_{p^l})| = u^{P^{\mathbb{F}_{p^l}}(y)}_{P^{\mathbb{F}_{p^l}}(x)} = p^{ls}(p^{l(h-s-1)} + \ldots + p^l + 1),$$

so using Lemma 1.15 we have that $\chi(X(\mathbb{C})) \neq 0$ which means that there is a monomorphism $P^{\mathbb{C}}(x) \to P^{\mathbb{C}}(y)$. □

Putting together all the pieces from above, we obtain our first main result:

Theorem 3.1. *Consider two fields k, k'. Then $\mu(P^k(x)) = \mu(P^{k'}(x))$, moreover, we have a GR inclusion $P^k(x) \to P^k(y)$ if and only if we have a GR inclusion $P^{k'}(x) \to P^{k'}(y)$.*

Proof. By Proposition 3.1 it is enough to consider the case when k has characteristic 0 and k' characteristic p. We know already that $\mu(P^k(x)) \leq \mu(P^{k'}(x))$. Conversely, by Proposition 3.1 and Proposition 3.3 we have $\mu(P^{k'}(x)) = \mu(P^{\mathbb{F}_p}(x)) \leq \mu(P^{\mathbb{C}}(x)) = \mu(P^{\mathbb{Q}}(x)) = \mu(P^k(x))$. The second part of the statement follows in the same manner as in the proof of Proposition 3.1. □

We consider now regular homogeneous indecomposable modules of dimension δ denoted simply by $R^k(a)$ (instead of $R^k(1,a)$), with $a \in \mathbb{H}_k(k)$. We will prove that the GR measure of $R^k(a)$ and also the GR inclusions in $R^k(a)$ do not depend on a and on the field k.

For the proof of the second main theorem we need the following result from [13].

Lemma 3.4. ([13]) *If P is a preprojective indecomposable and $R^k(a)$ is a homogeneous module of dimension, δ then $\mu(P) < \mu(R^k(a))$. Moreover, if $|P| < |\delta|$ then there is a monomorphism $P \to tR^k(a)$ for some $t \in \mathbb{N}^*$, so P is cogenerated by $R^k(a)$.*

Theorem 3.2. *Consider the fields k, k' and the points $a \in \mathbb{H}_k(k)$, $a' \in \mathbb{H}_{k'}(k')$. Then we have $\mu(R^k(a)) = \mu(R^{k'}(a'))$. Moreover, we have a GR inclusion $P^k(x) \to R^k(a)$ if and only if we have a GR inclusion $P^{k'}(x) \to R^{k'}(a')$.*

Proof. Suppose that $\mu(R^k(a)) < \mu(R^{k'}(a'))$ and denote by $P^k(x)$ and ${}^{k'}P(x')$ the GR submodules of $R^k(a)$ and $R^{k'}(a')$. On one hand we have that $\mu(P^k(x)) \cup \{|\delta|\} < \mu(P^{k'}(x')) \cup \{|\delta|\}$ which implies $\mu(P^k(x)) < \mu(P^{k'}(x'))$. On the other hand using Theorem 3.1 and the proposition above we get $\mu(P^{k'}(x')) = \mu(P^k(x')) < \mu(R^k(a)) = \mu(P^k(x)) \cup \{|\delta|\}$, a contradiction, since all the lengths in $\mu(P^k(x))$ and $\mu(P^{k'}(x'))$ are smaller then $|\delta|$.

Suppose now that we have a GR inclusion $P^{k'}(x) \to R^{k'}(a')$. This means, using the results above, that $\mu(R^k(a)) = \mu(R^{k'}(a')) = \mu(P^{k'}(x)) \cup \{|\delta|\} = \mu(P^k(x)) \cup \{|\delta|\}$. So one can see that $\mu(R^k(a))$ starts with $\mu(P^k(x))$.

Since $|P^k(x)| = |P^{k'}(x)| < |\delta|$, it follows by Lemma 3.4 that there is a monomorphism $P^k(x) \to tR^k(a)$ for some $t \in \mathbb{N}^*$. But then using Proposition 1.9 (c) it follows that there is a monomorphism $P^k(x) \to R^k(a)$. However, $\mu(R^k(a)) = \mu(P^k(x)) \cup \{|\delta|\}$, so this monomorphism is a GR inclusion. □

The following corollary clarifies the form of the Ringel-Hall polynomial corresponding to the GR inclusion $P^k(x) \to R^k(a)$.

Corollary 3.3. *Let k be a finite field with q elements. Then $u^{R^k(a)}_{P^k(x)} = F^{\delta}_{\delta - x\, x}(q) = \frac{q^{-\partial x} - q^s}{q-1}$ where $-\partial x = \langle x, \delta \rangle > s = \dim_k \operatorname{Sing}(P^k(x), R^k(a))$ are field independent and independent of a.*

Proof. Since we have a GR inclusion $P^k(x) \to R^k(a)$ then the factor is indecomposable, so it is isomorphic to $I^k(\delta - x)$. By Theorem 1.1 there is a rational polynomial $F^{\delta}_{\delta - x\, x}$ such that $F^{\delta}_{\delta - x\, x}(q) = F^{R^k(a)}_{I^k(\delta - x) P^k(x)}$. Since by the previous theorem the GR inclusions in $R^k(a)$ do not depend on a and on the field k, we obtain using Proposition 1.9 (f) that for q big enough $F^{\delta}_{\delta-x\,x}(q) = \frac{q^{-\partial x}-q^{s_k}}{q-1}$ (with $s_k < -\partial x$), which means that there is a value s such that for infinitely many values q we have $F^{\delta}_{\delta-x\,x}(q) = \frac{q^{-\partial x}-q^{s}}{q-1}$. But then $F^{\delta}_{\delta-x\,x} = \frac{X^{-\partial x}-X^{s}}{X-1}$. □

3.2 The defect of GR submodules in homogeneous modules of dimension δ

As an application of the results from the previous sections, using Ringel's idea from [45] we can prove the main result from [15] in a more general setting: the result is valid also for the case $\widetilde{\mathbb{E}}_8$ (this case is missing from [15]) and the base field k is arbitrary (in [15] k is algebraically closed).

Theorem 3.4. *Let Q be a tame quiver with minimal radical vector δ. If R is a homogeneous module with dimension δ and P a GR submodule, then P has defect -1. As a consequence the pair $(R/P, P)$ is a Kronecker pair.*

Proof. Since in the case $\widetilde{\mathbb{A}}_m$ the defect of a preprojective indecomposables is always -1, we suppose that Q is not of type $\widetilde{\mathbb{A}}_m$. We have $P = P^k(x)$, where x is a positive real root with $\partial x < 0$. Using Theorem 3.2 it follows that $P^k(x') \to R^{k'}(a')$ is a GR inclusion for any field k' and any point $a' \in \mathbb{H}_{k'}(k')$. By Corollary 3.3 this means that $F^{\delta}_{\delta-x x} = \frac{X^{-\partial x}-X^{s}}{X-1}$. But we also know from Theorem 2.5 that $F^{\delta}_{\delta-x x} = f_{-\partial x}$. Comparing the polynomials one can see that $\partial x = -1$.

Since R/P is indecomposable preinjective of defect 1, the second assertion follows using the observations in Section 1.10. □

Using Section 1.10, note that if $P = P^k(x)$ is a GR submodule in a homogeneous module R of dimension δ, then $R/P = I^k(\delta - x)$ and the pair $(I^k(\delta - x), P^k(x))$ exists and remains a Kronecker pair even in the case when the field has 2 elements and the quiver is not of type $\widetilde{\mathbb{A}}_m$ (so we do not have homogeneous modules of dimension δ).

Chapter 4

Cardinalities of Kronecker quiver Grassmannians

Let K be the Kronecker quiver (see Section 1.5), k a finite field with q elements and $\mathcal{H}(kK)$ the rational Ringel-Hall algebra of the Kronecker algebra (see Section 1.8).

For any module $M \in \text{mod-}kK$, and any $\underline{e} = (a, b)$ in $\mathbb{N}^2$, we denote by $Gr_{\underline{e}}(M)_k$ the Grassmannian variety of submodules of M with dimension vector $\underline{e}$:

$$Gr_{\underline{e}}(M)_k = \{N \in \text{mod-}kK | N \leq M, \underline{\dim}(N) = \underline{e}\}.$$

Then using the Ringel-Hall numbers F^M_{XY} we have that

$$|Gr_{\underline{e}}(M)_k| = \sum_{\substack{[X],[Y] \\ \underline{\dim} Y = \underline{e}}} F^M_{XY}.$$

The Grassmannian cardinalities above play an important role in the theory of cluster algebras and quantized cluster algebras (see [20, 21, 6, 5, 11]).

For M indecomposable (excepting some regular ones) $Gr_{\underline{e}}(M)_k$ is the set of k-points of a corresponding variety $Gr_{\underline{e}}(M)$ defined over $\mathbb{Z}$. Denote by $Gr_{\underline{e}}(M)_{\mathbb{C}}$ the set of $\mathbb{C}$-points of this variety. It is known (see Lemma 1.15) that if there exists a polynomial $p_{\underline{e},M}$ with integral coefficients such that $|Gr_{\underline{e}}(M)_k| = p_{\underline{e},M}(q)$ for infinitely many prime powers q, then the Euler-Poincaré characteristic (with compact support) of $Gr_{\underline{e}}(M)_{\mathbb{C}}$ is given by $\chi(Gr_{\underline{e}}(M)_{\mathbb{C}}) = p_{\underline{e},M}(1)$.

In [12] Caldero and Zelevinsky describe explicit combinatorial formulas for the Euler-Poincaré characteristics $\chi(Gr_{\underline{e}}(M)_{\mathbb{C}})$ whenever M is indecomposable.

Using the Ringel-Hall algebra approach and reflection functors we obtain specific recursions for the Grassmannian cardinalities (see the Chapter 1). Combining these with the q-analogue of Nanjundiah's identity (see the preliminary chapter, Proposition 1.1), we deduce in the second section explicit combinatorial formulas for the cardinalities $|Gr_{\underline{e}}(M)_k|$ whenever M is indecomposable. These formulas imply directly the existence of polynomials $p_{\underline{e},M}$ with integral coefficients such that $|Gr_{\underline{e}}(M)_k| = p_{\underline{e},M}(q)$. We realize in this way a quantification of the formulas by Caldero and Zelevinsky. In the last section we show that our recursions provide an algorithm for computing the cardinality of every Kronecker quiver Grassmannian over a finite field. All these results were published in [55].

4.1 The recursions

Recall from Section 1.9, Remark 1.5 that mod-$kK\langle i\rangle$ is the full subcategory of modules not containing the simple module corresponding to the vertex i as a direct summand and $T^+ = S_1^+$, $T^- = S_2^-$ induce exact quasi-inverse equivalences between mod-$kK\langle 1\rangle$ and mod-$kK\langle 2\rangle$. Let $a, b \in \mathbb{Z}$. We introduce the following notations:

- for $M \in$ mod-kK
$$A_{a,b}^M := |Gr_{(a,b)}(M)_k| = \sum_{\substack{[X],[Y] \\ \underline{\dim} Y=(a,b)}} F_{XY}^M;$$
- for $M \in$ mod-$kK\langle 1\rangle$
$$B_{a,b}^M := \sum_{\substack{[X],[Y] \\ \underline{\dim} Y=(a,b) \\ X,Y \in \text{mod-}kK\langle 1\rangle}} F_{XY}^M;$$
- for $M \in$ mod-$kK\langle 2\rangle$
$$C_{a,b}^M := \sum_{\substack{[X],[Y] \\ \underline{\dim} Y=(a,b) \\ X,Y \in \text{mod-}kK\langle 2\rangle}} F_{XY}^M.$$

The sums $A_{a,b}^M$, $B_{a,b}^M$, $C_{a,b}^M$ are considered to be 0 if they are empty. In particular they are 0 if $a < 0$ or $b < 0$. Also notice that if $\underline{\dim} M = (m, n)$ then $A_{a,b}^M = B_{a,b}^M = C_{a,b}^M = 0$ for $a > m$ or $b > n$.

Proposition 4.1. *Suppose that up to isomorphism $M = sS(1) \oplus M' \oplus tS(2)$ with $M' \in$ mod-$kK\langle 1\rangle \cap$ mod-$kK\langle 2\rangle$. Let $a, b \in \mathbb{Z}$, $l = a - b$ and $\underline{\dim} M = (m, n)$.*

(a) We have that
$$A_{a,b}^M = \sum_{c\in\mathbb{Z}} \binom{m-a+c}{c}_q B_{a-c,b}^{M'\oplus tS(2)} = \sum_{c\in\mathbb{Z}} \binom{m-a+c}{c}_q C_{a-l,b-l+c}^{T^+(M'\oplus tS(2))},$$
the sum being finite.

(b) We have that
$$A_{a,b}^M = \sum_{d\in\mathbb{Z}} \binom{b+d}{d}_q C_{a,b+d-t}^{sS(1)\oplus M'} = \sum_{d\in\mathbb{Z}} \binom{b+d}{d}_q B_{a+l-d+t,b+l}^{T^-(sS(1)\oplus M')},$$
the sum being finite.

Proof. (a) If $b < 0$ then trivially
$$A_{a,b}^M = \sum_{c\in\mathbb{Z}} \binom{m-a+c}{c}_q B_{a-c,b}^{M'\oplus tS(2)} = \sum_{c\in\mathbb{Z}} \binom{m-a+c}{c}_q C_{b,b-l+c}^{T^+(M'\oplus tS(2))} = 0.$$

If $a < 0$ we have $A_{a,b}^M = 0$. Then $\sum_{c\in\mathbb{Z}} \binom{m-a+c}{c}_q B_{a-c,b}^{M'\oplus tS(2)} = 0$ because $\binom{m-a+c}{c}_q = 0$ for $c < 0$, while for $c \geq 0$ we have $a - c < 0$ so $B_{a-c,b}^{M'\oplus tS(2)} = 0$. We also have $\sum_{c\in\mathbb{Z}} \binom{m-a+c}{c}_q C_{b,2b-a+c}^{T^+(M'\oplus tS(2))} = 0$ because

$\binom{m-a+c}{c}_q = 0$ for $c < 0$, while for $c \geq 0$ there is no $Y \in$ mod-$kK\langle 2\rangle$ with dimension $(b, 2b-a+c)$ (see Remark 1.5) so $C^{T^+(M'\oplus tS(2))}_{a-l,b-l+c} = 0$.

Consider now the case $a, b \geq 0$. Firstly notice that if $Y_c \in$ mod-$kK\langle 1\rangle$ then by Lemma 1.4, Lemma 1.8 and Remark 1.3 we have $[cS(1)][Y_c] = [cS(1) \oplus Y_c]$, so $F^Z_{cS(1)Y_c} = 1$ for $[Z] = [cS(1) \oplus Y_c]$ and $F^Z_{cS(1)Y_c} = 0$ in all the other cases. Then observe that $\sum_{[X]} F^Z_{X\ cS(1)} = |\{U \leq Z | U \cong cS(1)\}| = |Gr_c(k^{m-a+c})| = \binom{m-a+c}{c}_q$ for $Z = (k^{m-a+c}, k^{n-b}; f, g)$, since a submodule $U \leq Z$, $U \cong cS(1)$ has the form $U = (V, 0; 0, 0)$ with V subspace in k^{m-a+c} and $dim_k V = c$.

Using the associativity of the Hall product we obtain:

$$
\begin{aligned}
A^M_{a,b} &= \sum_{\substack{[X],[Y]\\ \underline{\dim} Y=(a,b)}} F^M_{XY} = \sum_{c\geq 0} \sum_{\substack{[X],[Y_c]\\ \underline{\dim} Y_c=(a-c,b)\\ Y_c\in \text{mod-}kK\langle 1\rangle}} F^M_{XcS(1)\oplus Y_c}\cdot 1 \\
&= \sum_{c\geq 0} \sum_{\substack{c[X],[Y_c],[Z]\\ \underline{\dim} Y_c=(a-c,b)\\ Y_c\in \text{mod-}kK\langle 1\rangle}} F^M_{XZ}\cdot F^Z_{cS(1)Y_c} = \sum_{c\geq 0} \sum_{\substack{[X],[Y_c],[Z]\\ \underline{\dim} Y_c=(a-c,b)\\ Y_c\in \text{mod-}kK\langle 1\rangle}} F^Z_{X\ cS(1)}\cdot F^M_{ZY_c} \\
&= \sum_{c\geq 0} \sum_{\substack{[Y_c],[Z]\\ \underline{\dim} Y_c=(a-c,b)\\ \underline{\dim} Z=(m-a+c,n-b)\\ Y_c\in \text{mod-}kK\langle 1\rangle}} \Big(\sum_{[X]} F^Z_{X\ cS(1)}\Big)\cdot F^M_{ZY_c} = \sum_{c\geq 0} \binom{m-a+c}{c}_q \sum_{\substack{[Y_c],[Z]\\ \underline{\dim} Y_c=(a-c,b)\\ Y_c\in \text{mod-}kK\langle 1\rangle}} F^M_{ZY_c} \\
&= \sum_{c\geq 0} \binom{m-a+c}{c}_q \sum_{\substack{[Y_c],[Z']\\ \underline{\dim} Y_c=(a-c,b)\\ Y_c,Z'\in \text{mod-}kK\langle 1\rangle}} F^{M'\oplus tS(2)}_{Z'Y_c} = \sum_{c\in\mathbb{Z}} \binom{m-a+c}{c}_q B^{M'\oplus tS(2)}_{a-c,b}.
\end{aligned}
$$

Here we have used the following: if $F^M_{ZY_c} = F^{sS(1)\oplus M'\oplus tS(2)}_{ZY_c} \neq 0$ then since $Y_c \in$ mod-$kK\langle 1\rangle$ it follows that Y_c embeds only in $M' \oplus tS(2)$ (see Lemma 1.4, Lemma 1.8), so $Z = sS(1) \oplus Z'$ with $0 \to Y_c \to M' \oplus tS(2) \to Z' \to 0$ exact, $Z' \in$ mod-$kK\langle 1\rangle$ (because $M' \oplus tS(2)$ does not project on $S(1)$) and in this way $F^{sS(1)\oplus M'\oplus tS(2)}_{ZY_c} = F^{M'\oplus tS(2)}_{Z'Y_c}$.

To prove the second equality we will use the functor T^+. By Lemma 1.10 we have that $B^{M'\oplus tS(2)}_{a-c,b} = C^{T^+(M'\oplus tS(2))}_{a-l,b-l+c}$.

(b) The proof is the dual of the previous proof.

□

We can state now the recursion theorem for the numbers $A^M_{a,b}$.

Theorem 4.1. *Suppose that up to isomorphism $M = sS(1) \oplus M' \oplus tS(2)$ with $M' \in$ mod-$kK\langle 1\rangle \cap$ mod-$kK\langle 2\rangle$. Let $a, b \in \mathbb{Z}$, $l = a - b$ and $\underline{\dim} M = (m, n)$. We have the following recursions:*

(a) $A^M_{a,b} = \sum_{c\in\mathbb{Z}} q^{c(b-l+c)} \binom{m-2b}{c}_q A^{T^+(M'\oplus tS(2))}_{a-l,b-l+c}$*, the sum being finite;*

(b) $A^M_{a,b} = \sum_{d\in\mathbb{Z}} q^{d(2m-n-a-l+d)} \binom{2a-2m+n}{d}_q A^{T^-(sS(1)\oplus M')}_{a+l-d+t,b+l}$*, the sum being finite.*

Proof. (a) Using the previous proposition and the fact that $T^+(M' \oplus tS(2)) \in \text{mod-}kK\langle 2\rangle$ we have

$$A^M_{a,b} = \sum_{c\in\mathbb{Z}} \binom{m-a+c}{c}_q C^{T^+(M'\oplus tS(2))}_{a-l,b-l+c},$$

$$A^{T^+(M'\oplus tS(2))}_{a-l,b-l+c} = \sum_{d\in\mathbb{Z}} \binom{b-l+c+d}{d}_q C^{T^+(M'\oplus tS(2))}_{a-l,b-l+c+d}.$$

Let $u = c+d$. Using Lemma 1.1 (c),

$$\begin{aligned}\sum_{c\in\mathbb{Z}} q^{c(b-l+c)}\binom{m-2b}{c}_q A^{T^+(M'\oplus tS(2))}_{a-l,b-l+c} &= \sum_{c,d\in\mathbb{Z}} q^{c(b-l+c)}\binom{m-2b}{c}_q\binom{b-l+c+d}{d}_q C^{T^+(M'\oplus tS(2))}_{a-l,b-l+c+d}\\ &= \sum_{u\in\mathbb{Z}}(\sum_{c\in\mathbb{Z}} q^{c(b-l+c)}\binom{m-2b}{c}_q\binom{b-l+u}{u-c}_q)C^{T^+(M'\oplus tS(2))}_{a-l,b-l+u}\\ &= \sum_{u\in\mathbb{Z}}\binom{m-a+u}{u}_q C^{T^+(M'\oplus tS(2))}_{a-l,b-l+u} = A^M_{a,b}.\end{aligned}$$

(b) Using the previous proposition, Remark 1.5 and the fact that $T^-(sS(1)\oplus M') \in \text{mod-}kK\langle 1\rangle$ we have

$$A^M_{a,b} = \sum_{d\in\mathbb{Z}}\binom{b+d}{d}_q B^{T^-(sS(1)\oplus M')}_{a+l-d+t,b+l},$$

$$A^{T^-(sS(1)\oplus M')}_{a+l-d+t,b+l} = \sum_{c\in\mathbb{Z}}\binom{2m-n-a-l+d+c}{c}_q B^{T^-(sS(1)\oplus M')}_{a+l-d+t-c,b+l}.$$

Let $u = c+d$. Using Lemma 1.1 (c),

$$\begin{aligned}&\sum_{d\in\mathbb{Z}} q^{d(2m-n-a-l+d)}\binom{2a-2m+n}{d}_q A^{T^-(sS(1)\oplus M')}_{a+l-d+t,b+l} =\\ &= \sum_{c,d\in\mathbb{Z}} q^{d(2m-n-a-l+d)}\binom{2a-2m+n}{d}_q\binom{2m-n-a-l+d+c}{c}_q B^{T^-(sS(1)\oplus M')}_{a+l-d+t-c,b+l}\\ &= \sum_{u\in\mathbb{Z}}(\sum_{d\in\mathbb{Z}} q^{d(2m-n-a-l+d)}\binom{2a-2m+n}{d}_q\binom{2m-n-a-l+u}{u-d}_q B^{T^-(sS(1)\oplus M')}_{a+l-u+t,b+l}\\ &= \sum_{u\in\mathbb{Z}}\binom{b+u}{u}_q B^{T^-(sS(1)\oplus M')}_{a+l-u+t,b+l} = A^M_{a,b}.\end{aligned}$$

□

4.2 Formulas for the cardinalities $A^M_{a,b} = |Gr_{(a,b)}(M)_k|$ with M indecomposable

Let $n \in \mathbb{N}$, $a,b \in \mathbb{Z}$. Using the recurrences from the previous chapter we will provide closed formulas for $A^{P_n}_{a,b}$, $A^{I_n}_{a,b}$ and $A^{R(t,x)}_{a,b}$ (where P_n is indecomposable preprojective, I_n is indecomposable preinjective and $R(t,x)$ is indecomposable regular homogeneous with quasi-lenght t and with $x \in \mathbb{H}_k(k)$, thus of degree $\deg x = 1$).

Theorem 4.2. $A_{a,b}^{P_n} = |Gr_{(a,b)}(P_n)_k| = \begin{cases} 0 & \textit{for } a < 0 \textit{ or } b < 0 \\ 1 & \textit{for } a = b = 0 \\ \binom{n+1-b}{n+1-a}_q \binom{a-1}{a-b-1}_q & \textit{otherwise} \end{cases}.$

Remark 4.1. (a) Using the definitions and Lemma 1.1 (a) notice that $\binom{n+1-b}{n+1-a}_q \binom{a-1}{a-b-1}_q = 0$ for $0 < a \leq b$, for $a > n+1$, for $b > n$, for $b < 0 < a$.

(b) In the recursions from Theorem 4.1 we will have nontrivial Gaussian coefficients $\binom{a}{\ell}_q$ with $a < 0$. For example using Theorem 4.2, the previous remark and the recursion from Theorem 4.1 (a) we have that

$$\binom{3}{1}_q\binom{5}{1}_q = A_{6,4}^{P_6} = \textstyle\sum_{c\in\mathbb{Z}} q^{c(2+c)}\binom{-1}{c}_q A_{4,2+c}^{P_5} = \binom{-1}{0}_q A_{4,2}^{P_5} + q^3\binom{-1}{1}_q A_{4,3}^{P_5}$$
$$= \binom{-1}{0}_q\binom{4}{2}_q\binom{3}{1}_q + q^3\binom{-1}{1}_q\binom{3}{2}_q\binom{3}{0}_q = \binom{4}{2}_q\binom{3}{1}_q + q^3\left(-\frac{1}{q}\right)\binom{3}{2}_q\binom{3}{0}_q = \binom{3}{1}_q\binom{5}{1}_q.$$

Proof. Induction on n. For $n = 0$ we have that $A_{a,b}^{P_0} = 1$ when $(a,b) = (1,0)$ or $(a,b) = (0,0)$ and 0 otherwise so using the previous remark we can see that the formula is true.

Suppose now $n \geq 1$. Then trivially $A_{a,b}^{P_n} = 0$ for $a < 0$ or $b < 0$ and $A_{0,0}^{P_n} = 1$ so we only need to look at the case $a, b \geq 0$, $(a,b) \neq (0,0)$. Let $l = a - b$. Using Theorem 4.1 (a) we obtain the recursion

$$A_{a,b}^{P_n} = \sum_{c\in\mathbb{Z}} q^{c(b-l+c)}\binom{n-2b+1}{c}_q A_{a-l,b-l+c}^{P_{n-1}},$$

the sum being finite.

Using Remark 4.1 and the induction hypothesis notice that if $b > 0$ (and $a \geq 0$) then

$$A_{a-l,b-l+c}^{P_{n-1}} = A_{b,2b-a+c}^{P_{n-1}} = \binom{n-2b+a-c}{n-b}_q\binom{b-1}{a-b-c-1}_q,$$

so denoting by $u = a-b-c-1$, using the previous recursion and the q analogue of Nanjundiah's identity (Proposition 1.1) with the entries $p = a-b-1$, $m = n+1-a$, $\mu = n+1-b$ and $\nu = a-1$ we get:

$$\begin{aligned} A_{a,b}^{P_n} &= \sum_{c\in\mathbb{Z}} q^{c(b-l+c)}\binom{n-2b+1}{c}_q A_{a-l,b-l+c}^{P_{n-1}} \\ &= \sum_{c\in\mathbb{Z}} q^{c(2b-a+c)}\binom{n-2b+1}{c}_q\binom{n-2b+a-c}{n-b}_q\binom{b-1}{a-b-c-1}_q \\ &= \sum_{u\in\mathbb{Z}} q^{(a-b-u-1)(b-u-1)}\binom{n-2b+1}{a-b-u-1}_q\binom{n-b+1+u}{n-b}_q\binom{b-1}{u}_q = \binom{n+1-b}{n+1-a}_q\binom{a-1}{a-b-1}_q. \end{aligned}$$

If now $b = 0$ (and $n+1 \geq a > 0$) then trivially

$$A_{a,b}^{P_n} = \sum_{[X]} F_{X\,aS(1)}^{P_n} = \binom{n+1}{a}_q = \binom{n+1-0}{n+1-a}_q\binom{a-1}{a-0-1}_q.$$

If $b = 0$ and $a > n+1$ then trivially $A_{a,b}^{P_n} = 0$ (see Remark 4.1 (a)). □

Theorem 4.3. $A^{I_n}_{a,b} = |Gr_{(a,b)}(I_n)_k| = \begin{cases} 0 & \textit{for } a > n \textit{ or } b > n+1 \\ 1 & \textit{for } a = n,\ b = n+1 \\ \binom{n-b}{a-b}_q \binom{a+1}{b}_q & \textit{otherwise} \end{cases}.$

Remark 4.2. Using the definitions and Lemma 1.1 (a) notice that $\binom{n-b}{a-b}_q \binom{a+1}{b}_q = 0$ for $a < b$, for $a < 0$, for $b < 0$, for $a > n$ and $b < n+1$.

Proof. We just dualize Theorem 4.2. More precisely reversing the arrows in the Kronecker quiver we have that

$$A^{I_n}_{a,b} = |Gr_{(a,b)}(I_n)_k| = \sum_{\substack{[X],[Y] \\ \underline{\dim} Y = (a,b)}} F^{I_n}_{XY} = \sum_{\substack{[X],[Y] \\ \underline{\dim} X = (b,a)}} F^{P_n}_{XY}$$

$$= \sum_{\substack{[X],[Y] \\ \underline{\dim} Y = (n+1-b, n-a)}} F^{P_n}_{XY} = |Gr_{(n+1-b,n-a)}(P_n)_{\mathbb{F}_q}| = A^{P_n}_{n+1-b,n-a}.$$

□

Lemma 4.1. *Let $t \in \mathbb{N}^*$, $a, b \in \mathbb{Z}$ and $x, x' \in \mathbb{H}_k(k)$. Then we have:*

(a) $A^{R(t,x)}_{a,a} = 1$ *for* $0 \le a \le t$*;*

(b) $A^{R(t,x)}_{a,b} = 0$ *for* $0 \le a < b \le t$*;*

(c) $A^{R(t,x)}_{a,b} = A^{R(t,x')}_{a,b}$.

Proof. (a) Suppose $\underline{\dim} Y = (a,a)$, $a > 0$ (so the defect is 0) and Y embeds into $R(t,x)$. Then using Lemma 1.4, Lemma 1.8 and the uniseriality of the regulars one can see that Y must be of the form $R(t',x)$ with $0 < t' \le t$. So it follows that for $0 < a \le n$, we have $A^{R(t,x)}_{a,a} = F^{R(t,x)}_{R(t'',x)R(t',x)} = 1$. The rest of the statement follows easily.

(b) If for $0 \le a < b \le t$ $A^{R(t,x)}_{a,b} > 0$ this would mean that there is a module Y of dimension (a,b) which embeds into $R(t,x)$. But $a < b$ means that the defect $\partial Y > 0$, so Y must have a preinjective component. Using Lemma 1.4, Lemma 1.8 one can notice that we can't embed a preinjective into a regular module.

(c) Using Lemma 1.4, Lemma 1.8 and the uniseriality of regulars, observe that for $F^{R(t,x)}_{XY} \ne 0$ the modules X, Y can contain at most a single regular direct component which is of the form $R(t',x)$. Permuting the points of $\mathbb{H}_k(k)$ the assertion follows.

□

Theorem 4.4. *Let $t \in \mathbb{N}^*$, $a, b \in \mathbb{Z}$ and $x \in \mathbb{H}_k(k)$ (thus of degree 1). Then we have*

$$A^{R(t,x)}_{a,b} = |Gr_{(a,b)}(R(t,x))_k| = \begin{cases} 0 & \textit{for } a < 0 \textit{ or } b < 0 \\ \binom{t-b}{t-a}_q \binom{a}{a-b}_q & \textit{otherwise} \end{cases}.$$

Remark 4.3. Using the definitions and Lemma 1.1 (a) notice that $\binom{t-b}{t-a}_q\binom{a}{a-b}_q = 0$ for $a < b$, for $a > t$, for $b > t$ and $\binom{t-b}{t-a}_q\binom{a}{a-b}_q = 1$ for $0 \le a = b \le t$.

Proof. Using the previous remark observe that the formula is trivially true whenever $a < 0$ or $b < 0$ or $a > t$ or $b > t$. Also when $b = 0$ and $0 \le a \le t$ then trivially

$$A^{R(t,x)}_{a,0} = \sum_{[X]} F^{R(t,x)}_{X\ aS(1)} = \binom{t}{a}_q = \binom{t-0}{t-a}_q\binom{a}{a-0}_q.$$

Using Lemma 4.1 one can see that the formula is true in the cases $0 \le a = b \le t$ and $0 \le a < b \le t$.

So we only need to consider the case $0 < b < a \le t$. Let $l = a - b$. Using Theorem 4.1 (a) and Lemma 4.1 (c) we obtain the recursion

$$A^{R(t,x)}_{a,b} = \sum_{c\in\mathbb{Z}} q^{c(b-l+c)}\binom{t-2b}{c}_q A^{R(t,x)}_{a-l,b-l+c},$$

the sum being finite.

We proceed by induction on a. Using the recursion and the considerations above for $a = 2 \le t$ we have

$$A^{R(t,x)}_{2,1} = \sum_{c\in\mathbb{Z}} q^{c^2}\binom{t-2}{c}_q A^{R(t,x)}_{1,c} = \binom{t-2}{0}_q\binom{t}{1}_q + q\binom{t-2}{1}_q = \binom{t-1}{t-2}_q\binom{2}{1}_q.$$

Let now $3 \le a \le t$ and $0 < b < a$. Using Remark 4.3 and the induction hypothesis notice that

$$A^{R(t,x)}_{a-l,b-l+c} = A^{R(t,x)}_{b,2b-a+c} = \binom{t-2b+a-c}{t-b}_q\binom{b}{a-b-c}_q,$$

so denoting by $u = a-b-c$, using the previous recursion and Proposition 1.1 with the entries $p = a-b$, $m = t-a$, $\mu = t-b$ and $\nu = a$ we get:

$$\begin{aligned} A^{R(t,x)}_{a,b} &= \sum_{c\in\mathbb{Z}} q^{c(b-l+c)}\binom{t-2b}{c}_q A^{R(t,x)}_{a-l,b-l+c} \\ &= \sum_{c\in\mathbb{Z}} q^{c(2b-a+c)}\binom{t-2b}{c}_q\binom{t-2b+a-c}{t-b}_q\binom{b}{a-b-c}_q \\ &= \sum_{u\in\mathbb{Z}} q^{(a-b-u)(b-u)}\binom{t-2b}{a-b-u}_q\binom{t-b+u}{t-b}_q\binom{b}{u}_q = \binom{t-b}{t-a}_q\binom{a}{a-b}_q. \end{aligned}$$

□

We can see from the previous theorems that in the cases $M = P_n$, $M = I_n$ or $M = R(t,x)$ for $x \in \mathbb{H}_k(k)$, the cardinality $|Gr_{\underline{e}}(M)_k|$ is a polynomial with integral coefficients $p_{\underline{e},M}(q)$. Also notice that in these cases $Gr_{\underline{e}}(M)_k$ is the set of k-points of a corresponding variety $Gr_{\underline{e}}(M)$ defined over $\mathbb{Z}$ (since the matrices of these representations are given by zeros and ones). Denoting by $Gr_{\underline{e}}(M)_{\mathbb{C}}$ the set of $\mathbb{C}$-points of this variety it follows by Lemma 1.15 that $\chi(Gr_{\underline{e}}(M)_{\mathbb{C}}) = p_{\underline{e},M}(1)$.

For $k = \mathbb{C}$ we can consider the complex path algebra $\mathbb{C}K$ of the Kronecker quiver and the category mod-$\mathbb{C}K$. For $\underline{e} \in \mathbb{N}^2$ we denote by $Gr_{\underline{e}}(M(\mathbb{C}))$ the Grassmannian variety of all submodules of dimension $\underline{e}$ in $M(\mathbb{C})$. Then we will have $Gr_{\underline{e}}(M)_{\mathbb{C}} = Gr_{\underline{e}}(M(\mathbb{C}))$ and so $\chi(Gr_{\underline{e}}(M(\mathbb{C}))) = p_{\underline{e},M}(1)$.

Using that $\binom{a}{n}_1 = \binom{a}{n}$ we obtain:

Corollary 4.5 ([12]). *(a)* $\chi(Gr_{(a,b)}(P_n(\mathbb{C}))) = \begin{cases} 0 & \textit{for } a < 0 \textit{ or } b < 0 \\ 1 & \textit{for } a = b = 0 \\ \binom{n+1-b}{n+1-a}\binom{a-1}{a-b-1} & \textit{otherwise} \end{cases};$

(b) $\chi(Gr_{(a,b)}(I_n(\mathbb{C}))) = \begin{cases} 0 & \textit{for } a > n \textit{ or } b > n+1 \\ 1 & \textit{for } a = n,\, b = n+1 \\ \binom{n-b}{a-b}\binom{a+1}{b} & \textit{otherwise} \end{cases};$

(c) $\chi(Gr_{(a,b)}(R(t,x,\mathbb{C}))) = \begin{cases} 0 & \textit{for } a < 0 \textit{ or } b < 0 \\ \binom{t-b}{t-a}\binom{a}{a-b} & \textit{otherwise} \end{cases}$, *where* $x \in \mathbb{H}_{\mathbb{C}}(\mathbb{C}) = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

4.3 A recursive algorithm for the cardinalities $A^M_{a,b} = |Gr_{(a,b)}(M)_k|$ with M arbitrary

Let $M \in$ mod-kK arbitrary and suppose $\underline{\dim} M = (m,n)$. We know that up to isomorphism $M = P \oplus R \oplus I$, where P (respectively I, R) is a module with all its indecomposable components preprojective (respectively preinjective, regular). We also know that $A^M_{a,b} = 0$ for $a < 0$ or $b < 0$ or $a > m$ or $b > n$.

Applying the recursion from Theorem 4.1 (a) after a finite number of steps $A^{P\oplus R\oplus I}_{a,b}$ is reduced to a finite number of cardinalities of the form $A^{R'\oplus I'}_{a',b'}$. Applying the recursion from Theorem 4.1 (b) after a finite number of steps $A^{R'\oplus I'}_{a',b'}$ is reduced to a finite number of cardinalities of the form $A^{R''}_{a'',b''}$. Using the arguments from the proof of Lemma 4.1 (b) we can see that $A^{R''}_{a'',b''} = 0$ for $a'' < b''$ so applying the recursion from Theorem 4.1 (a) after a finite number of steps $A^{R''}_{a'',b''}$ with $a'' \geq b'' \geq 0$ is reduced to a finite number of cardinalities of the form $A^{R'''}_{a''',a'''}$.

Suppose $R''' = \oplus_{i=1}^m R(\lambda^i, x_i)$, where λ^i are partitions, $x_i \in \mathbf{P}^1(k)$ are pairwise different closed points with degree $\deg x_i$ and $\sum_{i=1}^m \deg x_i |\lambda^i| = n$ so $\underline{\dim} R''' = (n,n)$. Denote a''' simply by a and suppose $0 \leq a \leq n$. For partitions ν, μ, λ we consider $g^{\nu}_{\lambda\mu}(q^{\deg x}) = F^{R(\nu,x)}_{R(\lambda,x)R(\mu,x)}$, the classical Hall polynomial (see Section 1.7). We know that $g^{\nu}_{\lambda\mu} = g^{\nu}_{\mu\lambda}$ and $g^{\nu}_{\lambda\mu} = 0$ unless $|\nu| = |\mu| + |\lambda|$ and $\mu, \lambda \subseteq \nu$.

Using Lemma 1.4, Lemma 1.8 and Remark 1.3 we have that

$$[\oplus_{i=1}^m R(\lambda^i, x_i)][\oplus_{i=1}^m R(\mu^i, x_i)] = \prod_{i=1}^m [R(\lambda^i, x_i)][R(\mu^i, x_i)],$$

so

$$F^{\oplus_{i=1}^m R(\nu^i,x_i)}_{\oplus_{i=1}^m R(\lambda^i,x_i)\ \oplus_{i=1}^m R(\mu^i,x_i)} = \prod_{i=1}^m F^{R(\nu^i,x_i)}_{R(\lambda^i,x_i)R(\mu^i,x_i)}.$$

Using the considerations above and the arguments from the proof of Lemma 4.1 (a) we will have

$$A^{R'''}_{a,a} = \sum_{\substack{\lambda^i,\mu^i \subseteq \nu^i \\ \sum_{i=1}^m \deg x_i |\mu^i| = a \\ \sum_{i=1}^m \deg x_i |\lambda^i| = n-a}} F^{\oplus_{i=1}^m R(\nu^i,x_i)}_{\oplus_{i=1}^m R(\lambda^i,x_i)\ \oplus_{i=1}^m R(\mu^i,x_i)} = \sum_{\substack{\lambda^i,\mu^i \subseteq \nu^i \\ \sum_{i=1}^m \deg x_i |\mu^i| = a \\ \sum_{i=1}^m \deg x_i |\lambda^i| = n-a}} \prod_{i=1}^m g^{\nu^i}_{\lambda^i\mu^i}(q^{\deg x_i}).$$

Appendix A

Schofield pairs

In this part we present the list of Schofield pairs of canonically oriented tame quivers (see Section 1.10). Following [49] we denote by $\Delta(\widetilde{\mathbb{A}}_{p,q})$, $\Delta(\widetilde{\mathbb{D}}_m)$, $\Delta(\widetilde{\mathbb{E}}_6)$, $\Delta(\widetilde{\mathbb{E}}_7)$, $\Delta(\widetilde{\mathbb{E}}_8)$ the canonically oriented quivers.

For every considered canonically oriented quiver $\Delta(Q)$, the corresponding data about the Schofield pairs is to be found in the section with the title "Schofield pairs for the quiver $\Delta(Q) - \delta = \dots$" where δ is the minimal radical vector in the case of Q. Each section begins with a drawing of $\Delta(Q)$ to specify the node labeling and we also give the Cartan and Coxeter matrices denoted by $C_{\Delta(Q)}$, respectively $\Phi_{\Delta(Q)}$. Then there will be three subdivisions ("Schofield pairs associated to preprojective exceptional modules", "Schofield pairs associated to preinjective exceptional modules" and "Schofield pairs associated to regular exceptional modules") containing the drawing of the corresponding part of the Auslander–Reiten quiver and the computed list of Schofield pairs associated to preprojective, preinjective, respectively regular exceptionals. On the graphical representation of the AR quiver, blue arrows show the existence of a so-called irreducible monomorphism, while red arrows represent the existence of irreducible epimorphisms between suitable indecomposable modules (for details see [2]). In case of each indecomposable on the Auslander–Reiten quiver, we show its dimension vector, since this is the only information needed to determine the Schofield sequences.

For any given $M \in \text{mod-}kQ$ exceptional with $s(M) = s$ we enlist all the $s - 1$ Schofield pairs associated to M as

$$M: \big(Y_1, X_1\big),\ \big(Y_2, X_2\big), \dots \big(Y_{s-1}, X_{s-1}\big),$$

when there are no special pairs associated to M, respectively

$$M: \big(Y_1, X_1\big),\ \dots,\ \big(Y_{s-2}, X_{s-2}\big),\ \big(uI, (u+1)P\big),$$

when M is a preprojective indecomposable with $\partial M = -1$ and

$$M: \big(Y_1, X_1\big),\ \dots,\ \big(Y_{s-2}, X_{s-2}\big),\ \big((v+1)I, vP\big),$$

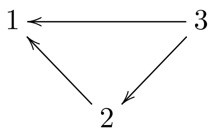
when M is a preinjective indecomposable with $\partial M = 1$. Note that for every M with $\partial M = \pm 1$, its associated special Schofield pair may be obtained using Corollary 1.4 (therefore, in the general formulas we just mark their presence). All special pairs are marked with teal color.

We group the modules M according to the vertices of the Auslander–Reiten quiver. Thus for every vertex $i \in Q_0$ we will have groups entitled "Modules of the form $P(n, i)$" enlisting Schofield pairs associated to families of preprojective exceptionals and "Modules of the form $I(n,i)$" with Schofield pairs associated to families of preinjective exceptionals. The regular exceptional modules are grouped by their non-homogeneous tubes. On the drawings depicting non-homogeneous tubes, the exceptional regulars are marked with green background color. We have marked with pink background the preprojectives and the preinjectives beyond which all Schofield sequences may be obtained by Auslander–Reiten translation (inverse translation in the case of preprojectives).

Although using Proposition 1.6 the Schofield pairs are completely described in the case of quiver of type $\mathbb{A}_m$, we compute and enlist concretely the Schofield pairs for the quivers $\Delta(\mathbb{A}_{1,2})$, $\Delta(\mathbb{A}_{1,3})$, $\Delta(\mathbb{A}_{1,4})$, $\Delta(\mathbb{A}_{2,2})$, $\Delta(\mathbb{A}_{2,3})$, $\Delta(\mathbb{A}_{2,4})$, $\Delta(\mathbb{A}_{3,3})$, $\Delta(\mathbb{A}_{3,4})$ and $\Delta(\mathbb{A}_{3,5})$. We also list the pairs for $\Delta(\mathbb{D}_4)$, $\Delta(\mathbb{D}_5)$ and $\Delta(\mathbb{D}_6)$ as examples before giving the general formulas for $\Delta(\mathbb{D}_m)$, $m \geq 4$. We end this part with the complete lists for $\Delta(\mathbb{E}_6)$, $\Delta(\mathbb{E}_7)$ and $\Delta(\mathbb{E}_8)$.

The computations were performed by a software implemented in the computer algebra system GAP (see [23]), which also generated the LaTeX source for this part (including the drawings of the various Auslander–Reiten quivers).

A.1 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{1,2})$ – $\delta = {}^{1}{}_{1}{}^{1}$



$$C_{\Delta(\widetilde{\mathbb{A}}_{1,2})} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{A}}_{1,2})} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1): -$

$P(1,1): \big(R_0^1(1), P(0,3)\big),\ \big(I(0,3), 2P(0,2)\big)$

$P(n,1): \big(R_0^{(n-1) \bmod 2+1}(1), P(n-1,3)\big),\ \big(uI, (u+1)P\big),\ n > 1$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$P(0,2): \big(R_0^1(1), P(0,1)\big)$

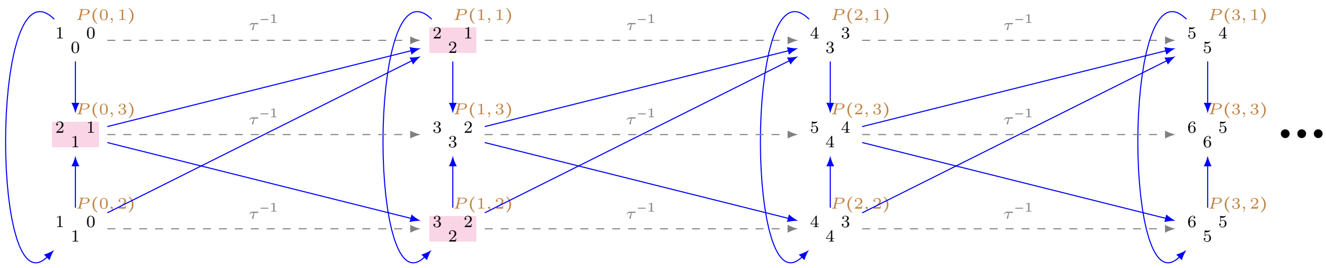
$P(1,2): \big(R_0^2(1), P(1,1)\big),\ \big(2I(0,2), 3P(0,1)\big)$

$P(n,2): \big(R_0^{n \bmod 2+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n > 1$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$P(0,3):\big(R_0^2(1),P(0,2)\big),\ \big(I(0,2),2P(0,1)\big)$
$P(n,3):\big(R_0^{(n+1)\text{ mod }2+1}(1),P(n,2)\big),\ \big(uI,(u+1)P\big),\ n>0$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1)=1$, for $n\geq 0$.

$I(0,1):\big(I(0,2),R_0^2(1)\big),\ \big(2I(0,3),P(0,2)\big)$
$I(n,1):\big(I(n,2),R_0^{(-n+1)\text{ mod }2+1}(1)\big),\ \big((v+1)I,vP\big),\ n>0$

Modules of the form $I(n,2)$

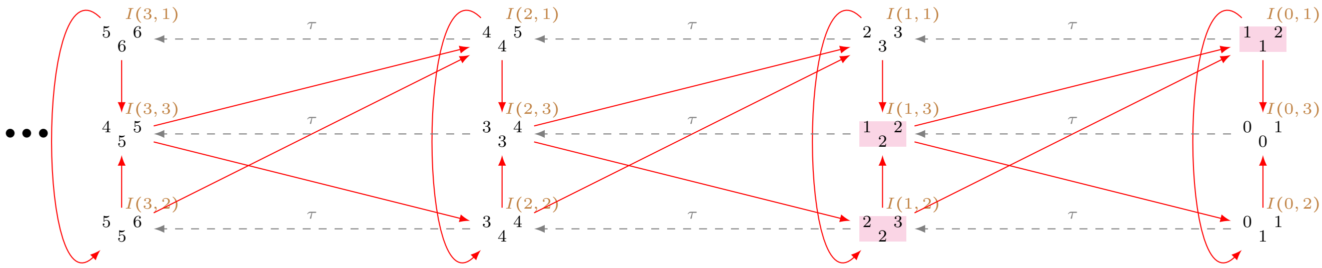
Defect: $\partial I(n,2)=1$, for $n\geq 0$.

$I(0,2):\big(I(0,3),R_0^1(1)\big)$
$I(1,2):\big(I(1,3),R_0^2(1)\big),\ \big(3I(0,3),2P(0,2)\big)$
$I(n,2):\big(I(n,3),R_0^{(-n+2)\text{ mod }2+1}(1)\big),\ \big((v+1)I,vP\big),\ n>1$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3)=1$, for $n\geq 0$.

$I(0,3):-$
$I(1,3):\big(I(0,1),R_0^1(1)\big),\ \big(2I(0,2),P(0,1)\big)$
$I(n,3):\big(I(n-1,1),R_0^{(-n+1)\text{ mod }2+1}(1)\big),\ \big((v+1)I,vP\big),\ n>1$

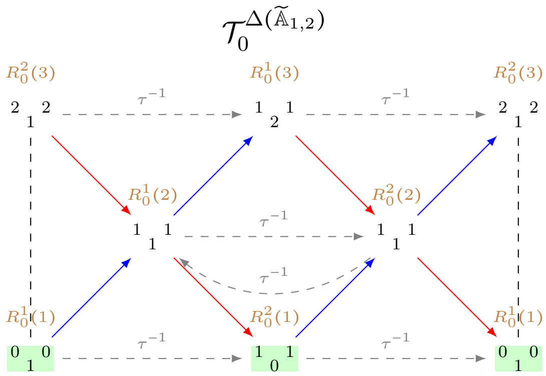


Schofield pairs associated to regular exceptional modules

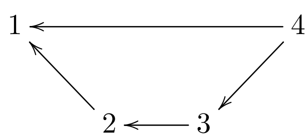
The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{1,2})}$

$R_0^1(1)$: –

$R_0^2(1)$: $\big(I(0,3), P(0,1)\big)$



A.2 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{1,3})$ – $\delta = {}^{1}{}_{1\ 1}{}^{1}$



$$C_{\Delta(\widetilde{\mathbb{A}}_{1,3})} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{A}}_{1,3})} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 2 \\ -2 & 1 & 0 & 2 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$$P(0,1): -$$
$$P(1,1): \big(R_0^2(2), P(0,3)\big),\ \big(R_0^3(1), P(0,4)\big),\ \big(I(0,3), 2P(0,2)\big)$$
$$P(n,1): \big(R_0^{n \bmod 3+1}(2), P(n-1,3)\big),\ \big(R_0^{(n+1) \bmod 3+1}(1), P(n-1,4)\big),\ \big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$$P(0,2): \big(R_0^3(1), P(0,1)\big)$$
$$P(1,2): \big(R_0^3(2), P(0,4)\big),\ \big(R_0^1(1), P(1,1)\big),\ \big(I(0,4), 2P(0,3)\big)$$
$$P(n,2): \big(R_0^{(n+1) \bmod 3+1}(2), P(n-1,4)\big),\ \big(R_0^{(n-1) \bmod 3+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,3)$

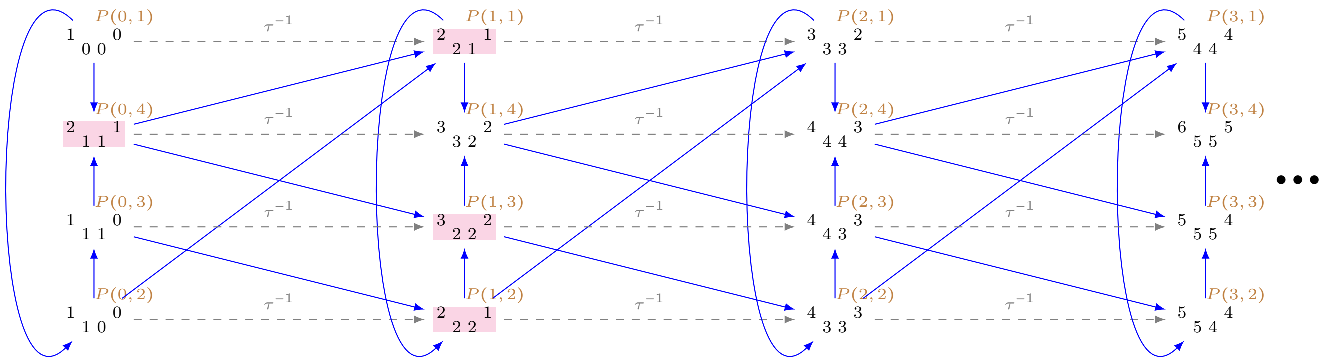
Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$$P(0,3): \big(R_0^3(2), P(0,1)\big),\ \big(R_0^1(1), P(0,2)\big)$$
$$P(1,3): \big(R_0^1(2), P(1,1)\big),\ \big(R_0^2(1), P(1,2)\big),\ \big(2I(0,2), 3P(0,1)\big)$$
$$P(n,3): \big(R_0^{(n-1) \bmod 3+1}(2), P(n,1)\big),\ \big(R_0^{n \bmod 3+1}(1), P(n,2)\big),\ \big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4) = -1$, for $n \geq 0$.

$$P(0,4): \big(R_0^1(2), P(0,2)\big),\ \big(R_0^2(1), P(0,3)\big),\ \big(I(0,2), 2P(0,1)\big)$$
$$P(n,4): \big(R_0^{n \bmod 3+1}(2), P(n,2)\big),\ \big(R_0^{(n+1) \bmod 3+1}(1), P(n,3)\big),\ \big(uI, (u+1)P\big),\ n > 0$$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$I(0,1) : \big(I(0,2), R_0^2(1)\big),\ \big(I(0,3), R_0^2(2)\big),\ \big(2I(0,4), P(0,3)\big)$
$I(n,1) : \big(I(n,2), R_0^{(-n+1) \bmod 3+1}(1)\big),\ \big(I(n,3), R_0^{(-n+1) \bmod 3+1}(2)\big),\ \big((v+1)I, vP\big),\ n > 0$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$I(0,2) : \big(I(0,3), R_0^3(1)\big),\ \big(I(0,4), R_0^3(2)\big)$
$I(1,2) : \big(I(1,3), R_0^2(1)\big),\ \big(I(1,4), R_0^2(2)\big),\ \big(3I(0,4), 2P(0,3)\big)$
$I(n,2) : \big(I(n,3), R_0^{(-n+2) \bmod 3+1}(1)\big),\ \big(I(n,4), R_0^{(-n+2) \bmod 3+1}(2)\big),\ \big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$I(0,3) : \big(I(0,4), R_0^1(1)\big)$
$I(1,3) : \big(I(0,1), R_0^3(2)\big),\ \big(I(1,4), R_0^3(1)\big),\ \big(2I(0,2), P(0,1)\big)$
$I(n,3) : \big(I(n-1,1), R_0^{(-n+3) \bmod 3+1}(2)\big),\ \big(I(n,4), R_0^{(-n+3) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 1$

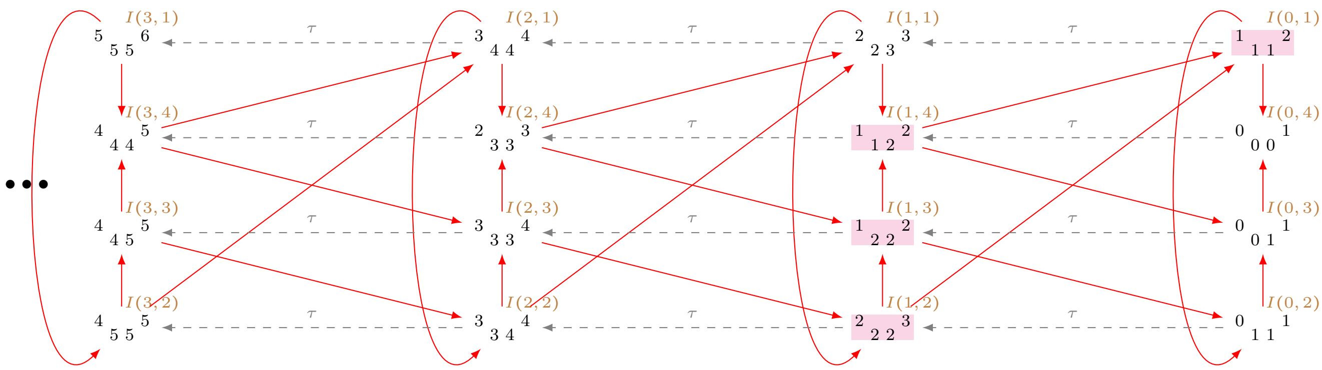
Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$I(0,4)$: –

$I(1,4) : \big(I(0,1), R_0^1(1)\big),\ \big(I(0,2), R_0^1(2)\big),\ \big(2I(0,3), P(0,2)\big)$

$I(n,4) : \big(I(n-1,1), R_0^{(-n+1) \bmod 3+1}(1)\big),\ \big(I(n-1,2), R_0^{(-n+1) \bmod 3+1}(2)\big),\ \big((v+1)I, vP\big),\ n > 1$



Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{1,3})}$

$R_0^1(1)$: –

$R_0^1(2) : \big(R_0^2(1), R_0^1(1)\big),\ \big(I(0,3), P(0,1)\big)$

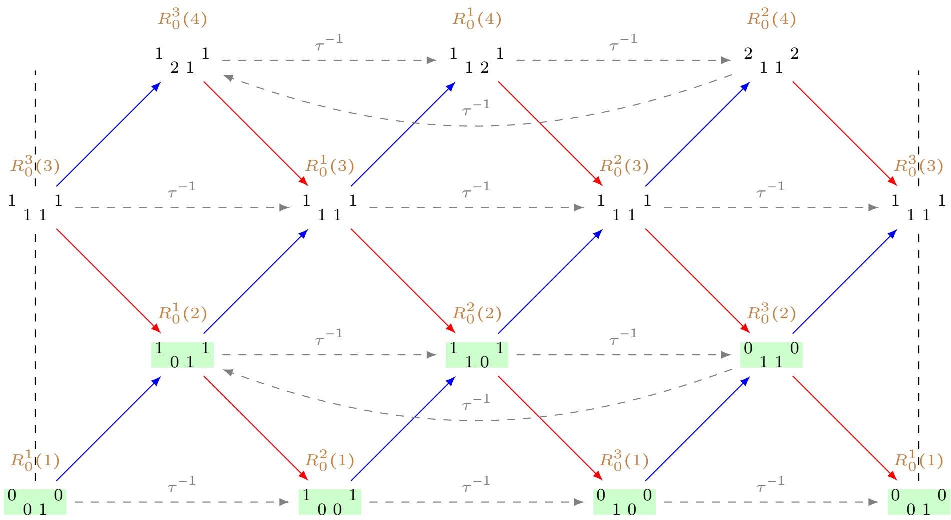
$R_0^2(1) : \big(I(0,4), P(0,1)\big)$

$R_0^2(2) : \big(R_0^3(1), R_0^2(1)\big),\ \big(I(0,4), P(0,2)\big)$

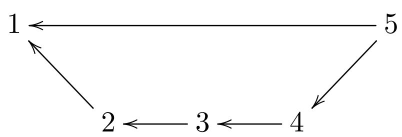
$R_0^3(1)$: –

$R_0^3(2) : \big(R_0^1(1), R_0^3(1)\big)$

$\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{1,3})}$



A.3 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{1,4})$ – $\delta = {}^{1}{}_{1\,1\,1}{}^{1}$



$$C_{\Delta(\widetilde{\mathbb{A}}_{1,4})} = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Phi_{\Delta(\widetilde{\mathbb{A}}_{1,4})} = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 2 \\ -2 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1): -$

$P(1,1): \big(R_0^1(3), P(0,3)\big),\ \big(R_0^2(2), P(0,4)\big),\ \big(R_0^3(1), P(0,5)\big),\ \big(I(0,3), 2P(0,2)\big)$

$P(n,1): \big(R_0^{(n-1) \bmod 4+1}(3), P(n-1,3)\big),\ \big(R_0^{n \bmod 4+1}(2), P(n-1,4)\big),\ \big(R_0^{(n+1) \bmod 4+1}(1), P(n-1,5)\big)$
$\big(uI, (u+1)P\big),\ n > 1$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$P(0,2): \big(R_0^3(1), P(0,1)\big)$

$P(1,2): \big(R_0^2(3), P(0,4)\big),\ \big(R_0^3(2), P(0,5)\big),\ \big(R_0^4(1), P(1,1)\big),\ \big(I(0,4), 2P(0,3)\big)$

$P(n,2): \big(R_0^{n \bmod 4+1}(3), P(n-1,4)\big),\ \big(R_0^{(n+1) \bmod 4+1}(2), P(n-1,5)\big),\ \big(R_0^{(n+2) \bmod 4+1}(1), P(n,1)\big)$

$\big(uI, (u+1)P\big),\ n > 1$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$$P(0,3) : \big(R_0^3(2), P(0,1)\big),\ \big(R_0^4(1), P(0,2)\big)$$
$$P(1,3) : \big(R_0^3(3), P(0,5)\big),\ \big(R_0^4(2), P(1,1)\big),\ \big(R_0^1(1), P(1,2)\big),\ \big(I(0,5), 2P(0,4)\big)$$
$$P(n,3) : \big(R_0^{(n+1) \bmod 4+1}(3), P(n-1,5)\big),\ \big(R_0^{(n+2) \bmod 4+1}(2), P(n,1)\big),\ \big(R_0^{(n-1) \bmod 4+1}(1), P(n,2)\big)$$
$$\big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,4)$

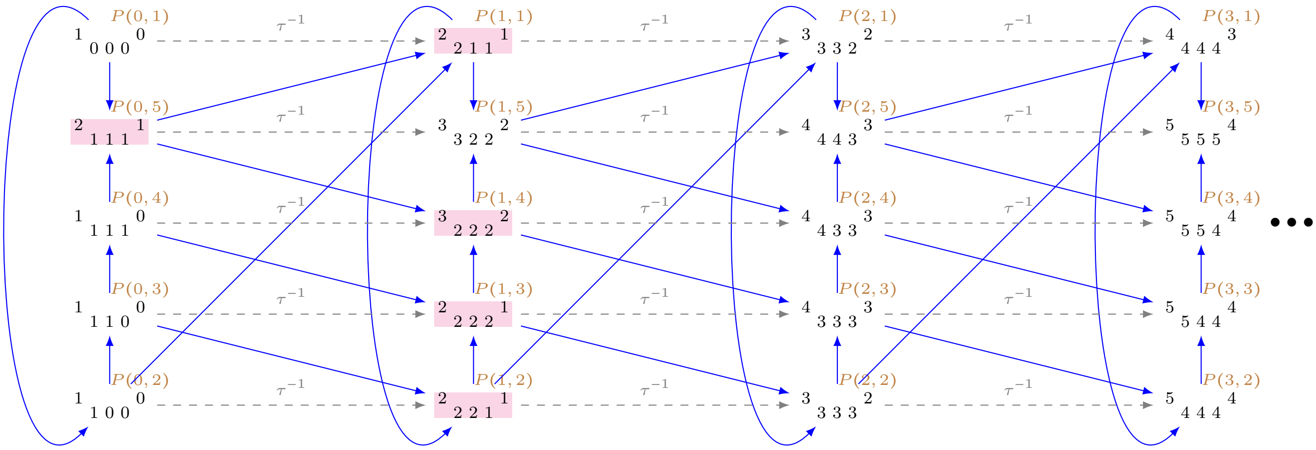
Defect: $\partial P(n,4) = -1$, for $n \geq 0$.

$$P(0,4) : \big(R_0^3(3), P(0,1)\big),\ \big(R_0^4(2), P(0,2)\big),\ \big(R_0^1(1), P(0,3)\big)$$
$$P(1,4) : \big(R_0^4(3), P(1,1)\big),\ \big(R_0^1(2), P(1,2)\big),\ \big(R_0^2(1), P(1,3)\big),\ \big(2I(0,2), 3P(0,1)\big)$$
$$P(n,4) : \big(R_0^{(n+2) \bmod 4+1}(3), P(n,1)\big),\ \big(R_0^{(n-1) \bmod 4+1}(2), P(n,2)\big),\ \big(R_0^{n \bmod 4+1}(1), P(n,3)\big)$$
$$\big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -1$, for $n \geq 0$.

$$P(0,5) : \big(R_0^4(3), P(0,2)\big),\ \big(R_0^1(2), P(0,3)\big),\ \big(R_0^2(1), P(0,4)\big),\ \big(I(0,2), 2P(0,1)\big)$$
$$P(n,5) : \big(R_0^{(n+3) \bmod 4+1}(3), P(n,2)\big),\ \big(R_0^{n \bmod 4+1}(2), P(n,3)\big),\ \big(R_0^{(n+1) \bmod 4+1}(1), P(n,4)\big)$$
$$\big(uI, (u+1)P\big),\ n > 0$$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$I(0,1) : \big(I(0,2), R_0^2(1)\big),\ \big(I(0,3), R_0^2(2)\big),\ \big(I(0,4), R_0^2(3)\big),\ \big(2I(0,5), P(0,4)\big)$

$I(n,1) : \big(I(n,2), R_0^{(-n+1) \bmod 4+1}(1)\big),\ \big(I(n,3), R_0^{(-n+1) \bmod 4+1}(2)\big),\ \big(I(n,4), R_0^{(-n+1) \bmod 4+1}(3)\big)$
$\big((v+1)I, vP\big),\ n > 0$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$I(0,2) : \big(I(0,3), R_0^3(1)\big),\ \big(I(0,4), R_0^3(2)\big),\ \big(I(0,5), R_0^3(3)\big)$

$I(1,2) : \big(I(1,3), R_0^2(1)\big),\ \big(I(1,4), R_0^2(2)\big),\ \big(I(1,5), R_0^2(3)\big),\ \big(3I(0,5), 2P(0,4)\big)$

$I(n,2) : \big(I(n,3), R_0^{(-n+2) \bmod 4+1}(1)\big),\ \big(I(n,4), R_0^{(-n+2) \bmod 4+1}(2)\big),\ \big(I(n,5), R_0^{(-n+2) \bmod 4+1}(3)\big)$
$\big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$I(0,3) : \big(I(0,4), R_0^4(1)\big),\ \big(I(0,5), R_0^4(2)\big)$

$I(1,3) : \big(I(0,1), R_0^3(3)\big),\ \big(I(1,4), R_0^3(1)\big),\ \big(I(1,5), R_0^3(2)\big),\ \big(2I(0,2), P(0,1)\big)$

$I(n,3) : \big(I(n-1,1), R_0^{(-n+3) \bmod 4+1}(3)\big),\ \big(I(n,4), R_0^{(-n+3) \bmod 4+1}(1)\big),\ \big(I(n,5), R_0^{(-n+3) \bmod 4+1}(2)\big)$
$\big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,4)$

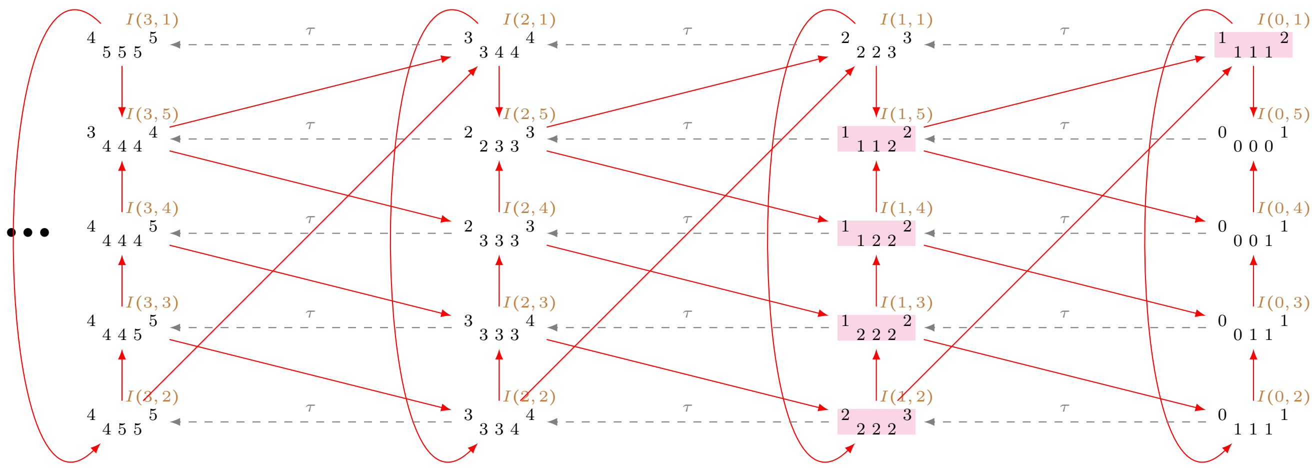
Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$I(0,4) : \big(I(0,5), R_0^1(1)\big)$
$I(1,4) : \big(I(0,1), R_0^4(2)\big),\ \big(I(0,2), R_0^4(3)\big),\ \big(I(1,5), R_0^4(1)\big),\ \big(2I(0,3), P(0,2)\big)$
$I(n,4) : \big(I(n-1,1), R_0^{(-n+4) \bmod 4+1}(2)\big),\ \big(I(n-1,2), R_0^{(-n+4) \bmod 4+1}(3)\big),\ \big(I(n,5), R_0^{(-n+4) \bmod 4+1}(1)\big)$
$\big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 1$, for $n \geq 0$.

$I(0,5) : -$
$I(1,5) : \big(I(0,1), R_0^1(1)\big),\ \big(I(0,2), R_0^1(2)\big),\ \big(I(0,3), R_0^1(3)\big),\ \big(2I(0,4), P(0,3)\big)$
$I(n,5) : \big(I(n-1,1), R_0^{(-n+1) \bmod 4+1}(1)\big),\ \big(I(n-1,2), R_0^{(-n+1) \bmod 4+1}(2)\big),\ \big(I(n-1,3), R_0^{(-n+1) \bmod 4+1}(3)\big)$
$\big((v+1)I, vP\big),\ n > 1$



Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{1,4})}$

$R_0^1(1) : -$

$R_0^1(2) : \big(R_0^2(1), R_0^1(1)\big),\ \big(I(0,4), P(0,1)\big)$

$R_0^2(1) : \big(I(0,5), P(0,1)\big)$

$R_0^2(2) : \big(R_0^3(1), R_0^2(1)\big),\ \big(I(0,5), P(0,2)\big)$

$R_0^3(1) : -$

$R_0^3(2) : \big(R_0^4(1), R_0^3(1)\big)$

$R_0^4(1) : -$

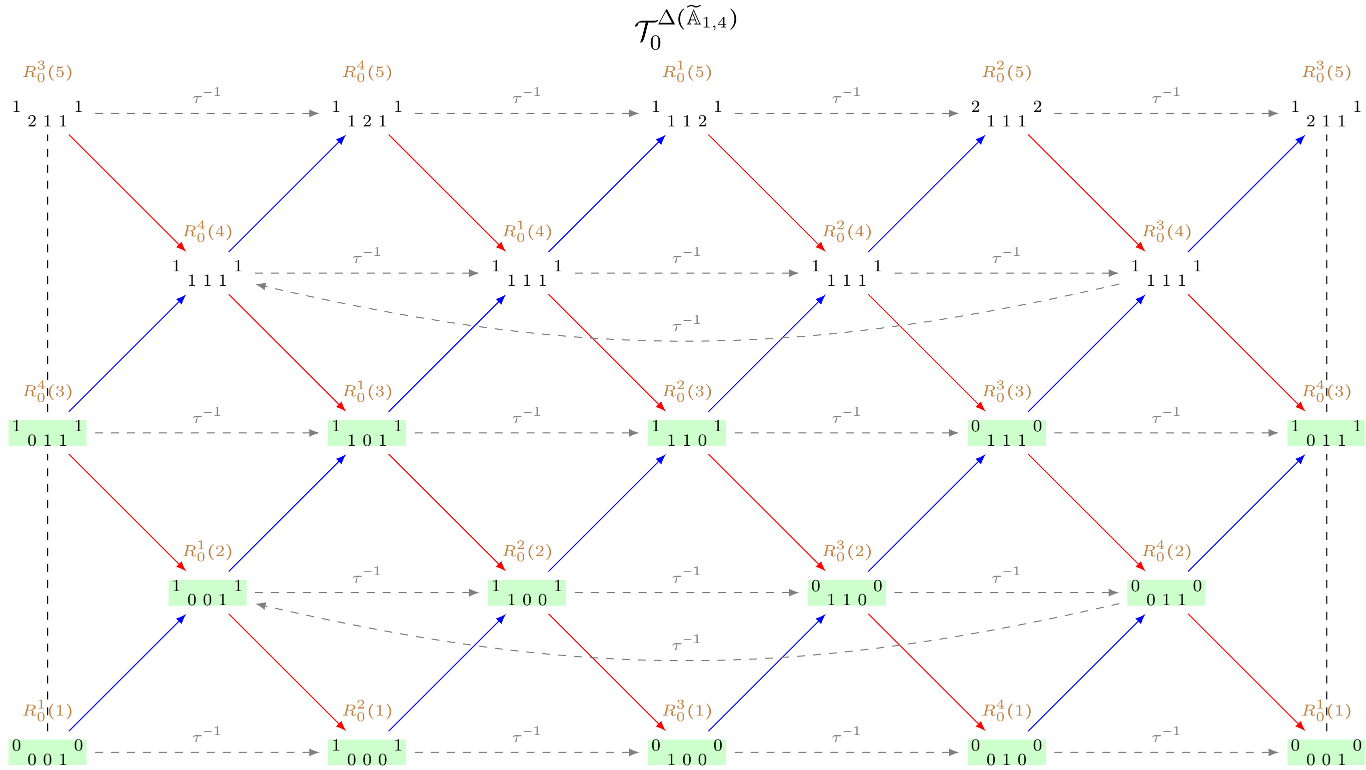
$R_0^4(2) : \big(R_0^1(1), R_0^4(1)\big)$

$R_0^4(3) : \big(R_0^1(2), R_0^4(1)\big),\ \big(R_0^2(1), R_0^4(2)\big),\ \big(I(0,3), P(0,1)\big)$

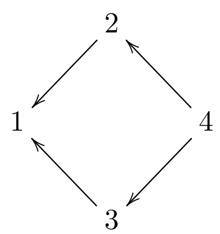
$R_0^1(3) : \big(R_0^2(2), R_0^1(1)\big),\ \big(R_0^3(1), R_0^1(2)\big),\ \big(I(0,4), P(0,2)\big)$

$R_0^2(3) : \big(R_0^3(2), R_0^2(1)\big),\ \big(R_0^4(1), R_0^2(2)\big),\ \big(I(0,5), P(0,3)\big)$

$R_0^3(3) : \big(R_0^4(2), R_0^3(1)\big),\ \big(R_0^1(1), R_0^3(2)\big)$



A.4 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{2,2})$ – $\delta = {}_1 {}^{1}_{1} {}_1$



$$C_{\Delta(\widetilde{\mathbb{A}}_{2,2})} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{A}}_{2,2})} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -2 & 1 & 1 & 1 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1)$: –
$P(1,1) : \big(R_0^1(1), P(0,2)\big),\ \big(R_\infty^1(1), P(0,3)\big)$
$P(2,1) : \big(R_0^2(1), P(1,2)\big),\ \big(R_\infty^2(1), P(1,3)\big),\ \big(2I(1,4), 3P(0,1)\big)$
$P(n,1) : \big(R_0^{(n-1) \bmod 2+1}(1), P(n-1,2)\big),\ \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,3)\big),\ \big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$$P(0,2) : \big(R^1_\infty(1), P(0,1)\big)$$
$$P(1,2) : \big(R^1_0(1), P(0,4)\big),\ \big(R^2_\infty(1), P(1,1)\big),\ \big(I(0,2), 2P(0,3)\big)$$
$$P(n,2) : \big(R_0^{(n-1) \bmod 2+1}(1), P(n-1,4)\big),\ \big(R_\infty^{n \bmod 2+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,3)$

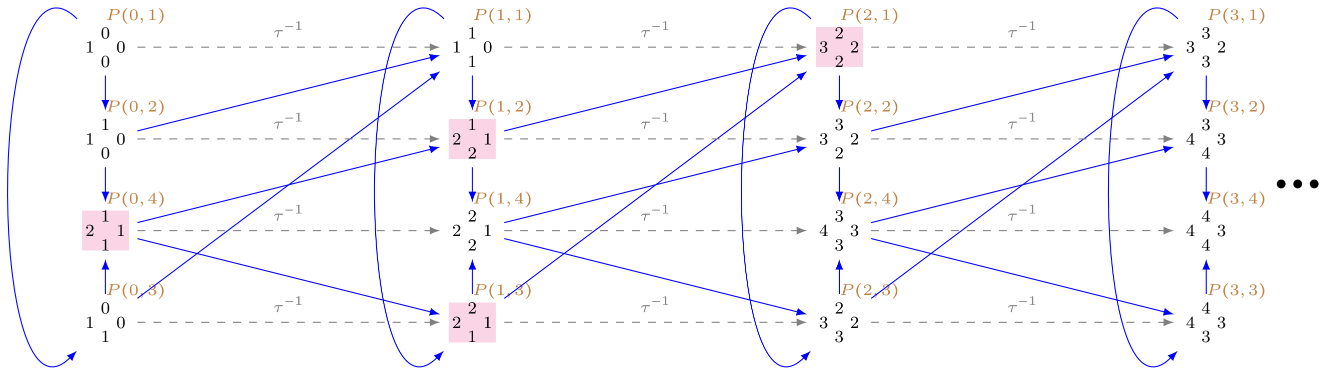
Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$$P(0,3) : \big(R^1_0(1), P(0,1)\big)$$
$$P(1,3) : \big(R^1_\infty(1), P(0,4)\big),\ \big(R^2_0(1), P(1,1)\big),\ \big(I(0,3), 2P(0,2)\big)$$
$$P(n,3) : \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,4)\big),\ \big(R_0^{n \bmod 2+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4) = -1$, for $n \geq 0$.

$$P(0,4) : \big(R^2_\infty(1), P(0,2)\big),\ \big(R^2_0(1), P(0,3)\big),\ \big(I(1,4), 2P(0,1)\big)$$
$$P(n,4) : \big(R_\infty^{(n+1) \bmod 2+1}(1), P(n,2)\big),\ \big(R_0^{(n+1) \bmod 2+1}(1), P(n,3)\big),\ \big(uI, (u+1)P\big),\ n > 0$$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$I(0,1) : \big(I(0,2), R^2_\infty(1)\big),\ \big(I(0,3), R^2_0(1)\big),\ \big(2I(0,4), P(1,1)\big)$
$I(n,1) : \big(I(n,2), R^{(-n+1) \bmod 2+1}_\infty(1)\big),\ \big(I(n,3), R^{(-n+1) \bmod 2+1}_0(1)\big),\ \big((v+1)I, vP\big),\ n > 0$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$I(0,2) : \big(I(0,4), R^1_\infty(1)\big)$
$I(1,2) : \big(I(0,1), R^1_0(1)\big),\ \big(I(1,4), R^2_\infty(1)\big),\ \big(2I(0,3), P(0,2)\big)$
$I(n,2) : \big(I(n-1,1), R^{(-n+1) \bmod 2+1}_0(1)\big),\ \big(I(n,4), R^{(-n+2) \bmod 2+1}_\infty(1)\big),\ \big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,3)$

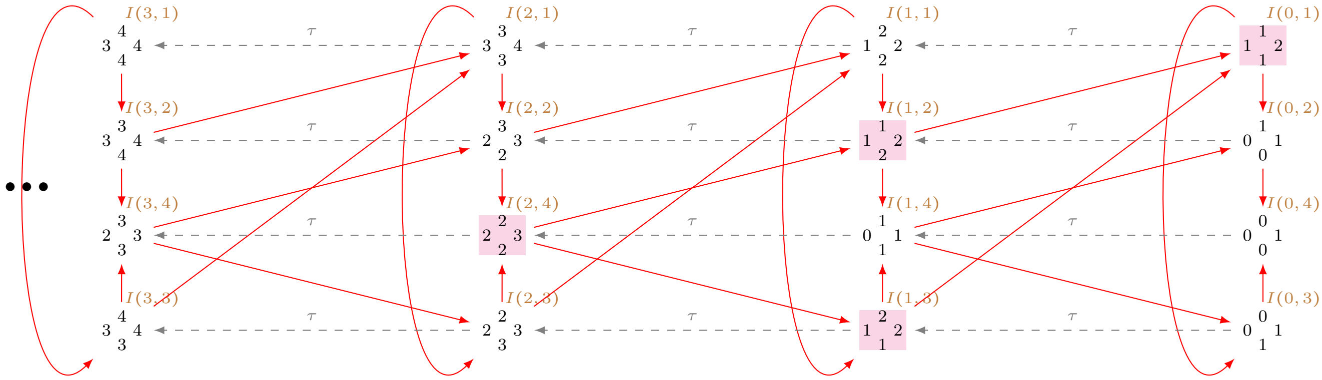
Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$I(0,3) : \big(I(0,4), R^1_0(1)\big)$
$I(1,3) : \big(I(0,1), R^1_\infty(1)\big),\ \big(I(1,4), R^2_0(1)\big),\ \big(2I(0,2), P(0,3)\big)$
$I(n,3) : \big(I(n-1,1), R^{(-n+1) \bmod 2+1}_\infty(1)\big),\ \big(I(n,4), R^{(-n+2) \bmod 2+1}_0(1)\big),\ \big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$I(0,4) : -$
$I(1,4) : \big(I(0,2), R^1_0(1)\big),\ \big(I(0,3), R^1_\infty(1)\big)$
$I(2,4) : \big(I(1,2), R^2_0(1)\big),\ \big(I(1,3), R^2_\infty(1)\big),\ \big(3I(0,4), 2P(1,1)\big)$
$I(n,4) : \big(I(n-1,2), R^{(-n+3) \bmod 2+1}_0(1)\big),\ \big(I(n-1,3), R^{(-n+3) \bmod 2+1}_\infty(1)\big),\ \big((v+1)I, vP\big),\ n > 2$

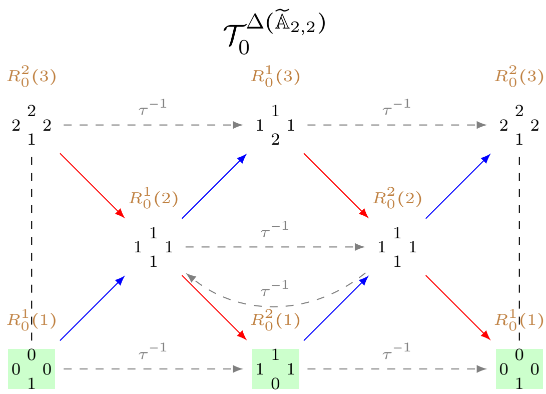


Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{2,2})}$

$R_0^1(1)$: –

$R_0^2(1)$: $\big(I(0,2), P(0,1)\big)$, $\big(I(0,4), P(0,2)\big)$

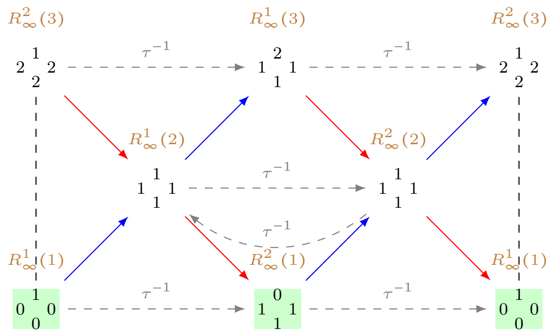


The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{2,2})}$

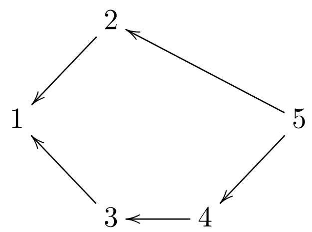
$R_\infty^1(1)$: –

$R_\infty^2(1)$: $\big(I(0,3), P(0,1)\big)$, $\big(I(0,4), P(0,3)\big)$

$\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{2,2})}$



A.5 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{2,3})$ – $\delta = {}_{1}\,{}^{1}_{1\,1}\,{}_{1}$



$$C_{\Delta(\widetilde{\mathbb{A}}_{2,3})} = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{A}}_{2,3})} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1)$: –
$P(1,1) : \big(R_0^3(1), P(0,2)\big),\ \big(R_\infty^1(1), P(0,3)\big)$
$P(2,1) : \big(R_0^3(2), P(0,5)\big),\ \big(R_0^1(1), P(1,2)\big),\ \big(R_\infty^2(1), P(1,3)\big),\ \big(I(0,2), 2P(0,4)\big)$
$P(n,1) : \big(R_0^{n \bmod 3+1}(2), P(n-2,5)\big),\ \big(R_0^{(n-2) \bmod 3+1}(1), P(n-1,2)\big),\ \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,3)\big)$
$\big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$P(0,2) : \big(R_\infty^1(1), P(0,1)\big)$
$P(1,2) : \big(R_0^2(2), P(0,4)\big),\ \big(R_0^3(1), P(0,5)\big),\ \big(R_\infty^2(1), P(1,1)\big),\ \big(I(1,5), 2P(0,3)\big)$
$P(n,2) : \big(R_0^{n \bmod 3+1}(2), P(n-1,4)\big),\ \big(R_0^{(n+1) \bmod 3+1}(1), P(n-1,5)\big),\ \big(R_\infty^{n \bmod 2+1}(1), P(n,1)\big)$
$\big(uI, (u+1)P\big),\ n > 1$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$$P(0,3) : \big(R_0^3(1), P(0,1)\big)$$
$$P(1,3) : \big(R_0^3(2), P(0,2)\big),\ \big(R_\infty^1(1), P(0,4)\big),\ \big(R_0^1(1), P(1,1)\big)$$
$$P(2,3) : \big(R_0^1(2), P(1,2)\big),\ \big(R_\infty^2(1), P(1,4)\big),\ \big(R_0^2(1), P(2,1)\big),\ \big(2I(1,4), 3P(0,1)\big)$$
$$P(n,3) : \big(R_0^{(n-2) \bmod 3+1}(2), P(n-1,2)\big),\ \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,4)\big),\ \big(R_0^{(n-1) \bmod 3+1}(1), P(n,1)\big)$$
$$\big(uI, (u+1)P\big),\ n > 2$$

Modules of the form $P(n,4)$

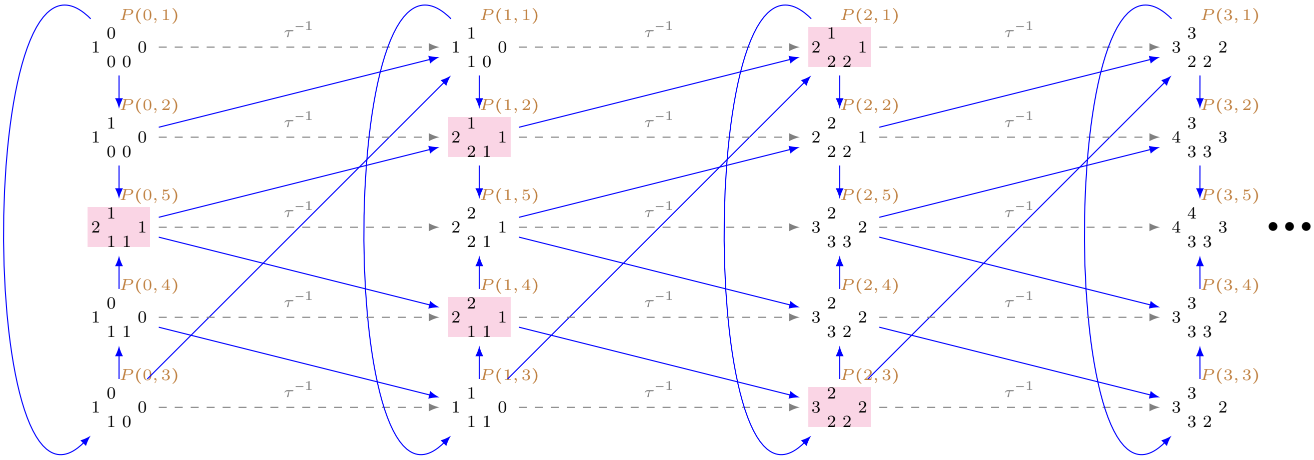
Defect: $\partial P(n,4) = -1$, for $n \geq 0$.

$$P(0,4) : \big(R_0^3(2), P(0,1)\big),\ \big(R_0^1(1), P(0,3)\big)$$
$$P(1,4) : \big(R_\infty^1(1), P(0,5)\big),\ \big(R_0^1(2), P(1,1)\big),\ \big(R_0^2(1), P(1,3)\big),\ \big(I(0,3), 2P(0,2)\big)$$
$$P(n,4) : \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,5)\big),\ \big(R_0^{(n-1) \bmod 3+1}(2), P(n,1)\big),\ \big(R_0^{n \bmod 3+1}(1), P(n,3)\big)$$
$$\big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -1$, for $n \geq 0$.

$$P(0,5) : \big(R_\infty^2(1), P(0,2)\big),\ \big(R_0^1(2), P(0,3)\big),\ \big(R_0^2(1), P(0,4)\big),\ \big(I(1,4), 2P(0,1)\big)$$
$$P(n,5) : \big(R_\infty^{(n+1) \bmod 2+1}(1), P(n,2)\big),\ \big(R_0^{n \bmod 3+1}(2), P(n,3)\big),\ \big(R_0^{(n+1) \bmod 3+1}(1), P(n,4)\big)$$
$$\big(uI, (u+1)P\big),\ n > 0$$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$$I(0,1) : \big(I(0,2), R_\infty^2(1)\big),\ \big(I(0,3), R_0^2(1)\big),\ \big(I(0,4), R_0^2(2)\big),\ \big(2I(0,5), P(1,3)\big)$$
$$I(n,1) : \big(I(n,2), R_\infty^{(-n+1) \bmod 2+1}(1)\big),\ \big(I(n,3), R_0^{(-n+1) \bmod 3+1}(1)\big),\ \big(I(n,4), R_0^{(-n+1) \bmod 3+1}(2)\big)$$
$$\big((v+1)I, vP\big),\ n > 0$$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$$I(0,2) : \big(I(0,5), R_\infty^1(1)\big)$$
$$I(1,2) : \big(I(0,1), R_0^1(1)\big),\ \big(I(0,3), R_0^1(2)\big),\ \big(I(1,5), R_\infty^2(1)\big),\ \big(2I(0,4), P(1,1)\big)$$
$$I(n,2) : \big(I(n-1,1), R_0^{(-n+1) \bmod 3+1}(1)\big),\ \big(I(n-1,3), R_0^{(-n+1) \bmod 3+1}(2)\big),\ \big(I(n,5), R_\infty^{(-n+2) \bmod 2+1}(1)\big)$$
$$\big((v+1)I, vP\big),\ n > 1$$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$$I(0,3) : \big(I(0,4), R_0^3(1)\big),\ \big(I(0,5), R_0^3(2)\big)$$

$I(1,3) : \big(I(0,1), R_\infty^1(1)\big),\ \big(I(1,4), R_0^2(1)\big),\ \big(I(1,5), R_0^2(2)\big),\ \big(2I(0,2), P(0,4)\big)$

$I(n,3) : \big(I(n-1,1), R_\infty^{(-n+1) \bmod 2+1}(1)\big),\ \big(I(n,4), R_0^{(-n+2) \bmod 3+1}(1)\big),\ \big(I(n,5), R_0^{(-n+2) \bmod 3+1}(2)\big)$
$\big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,4)$

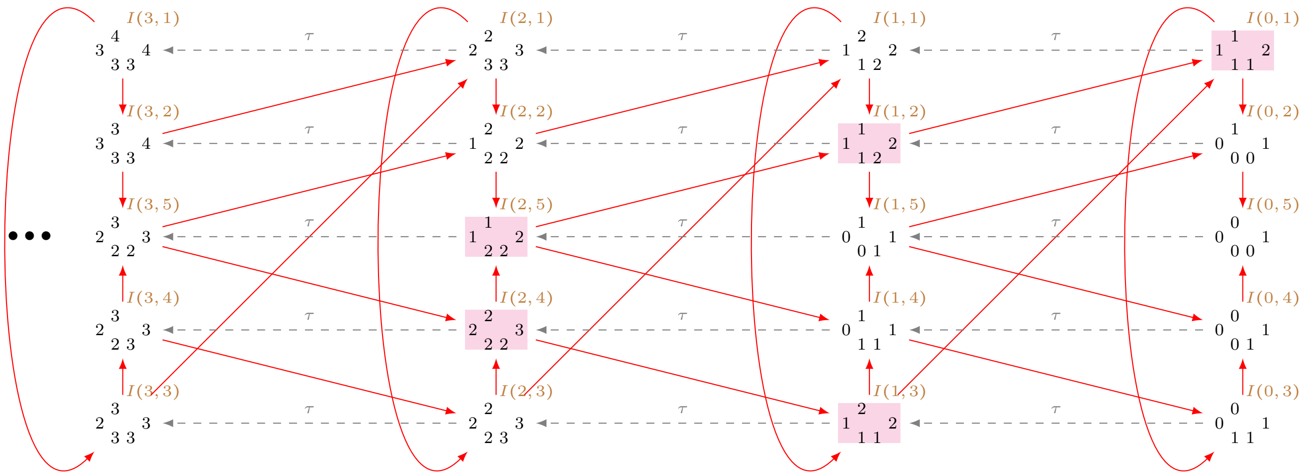
Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$I(0,4) : \big(I(0,5), R_0^1(1)\big)$
$I(1,4) : \big(I(0,2), R_0^3(2)\big),\ \big(I(0,3), R_\infty^1(1)\big),\ \big(I(1,5), R_0^3(1)\big)$
$I(2,4) : \big(I(1,2), R_0^2(2)\big),\ \big(I(1,3), R_\infty^2(1)\big),\ \big(I(2,5), R_0^2(1)\big),\ \big(3I(0,5), 2P(1,3)\big)$
$I(n,4) : \big(I(n-1,2), R_0^{(-n+3) \bmod 3+1}(2)\big),\ \big(I(n-1,3), R_\infty^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n,5), R_0^{(-n+3) \bmod 3+1}(1)\big)$
$\big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 1$, for $n \geq 0$.

$I(0,5) : -$
$I(1,5) : \big(I(0,2), R_0^1(1)\big),\ \big(I(0,4), R_\infty^1(1)\big)$
$I(2,5) : \big(I(0,1), R_0^3(2)\big),\ \big(I(1,2), R_0^3(1)\big),\ \big(I(1,4), R_\infty^2(1)\big),\ \big(2I(0,3), P(0,2)\big)$
$I(n,5) : \big(I(n-2,1), R_0^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,2), R_0^{(-n+4) \bmod 3+1}(1)\big),\ \big(I(n-1,4), R_\infty^{(-n+3) \bmod 2+1}(1)\big)$
$\big((v+1)I, vP\big),\ n > 2$



Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{2,3})}$

$R_0^1(1)$: –

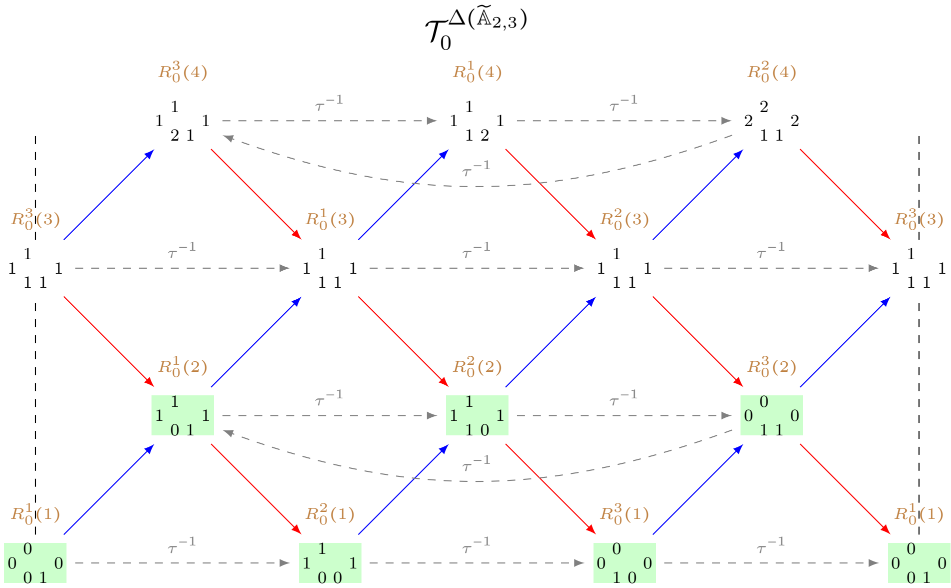
$R_0^1(2) : \big(R_0^2(1), R_0^1(1)\big),\ \big(I(1,5), P(0,1)\big),\ \big(I(0,4), P(0,2)\big)$

$R_0^2(1) : \big(I(0,2), P(0,1)\big),\ \big(I(0,5), P(0,2)\big)$

$R_0^2(2) : \big(R_0^3(1), R_0^2(1)\big),\ \big(I(0,2), P(0,3)\big),\ \big(I(0,5), P(1,1)\big)$

$R_0^3(1)$: –

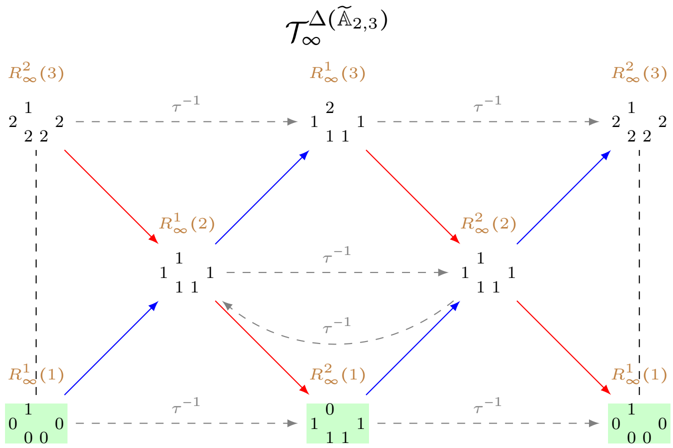
$R_0^3(2) : \big(R_0^1(1), R_0^3(1)\big)$



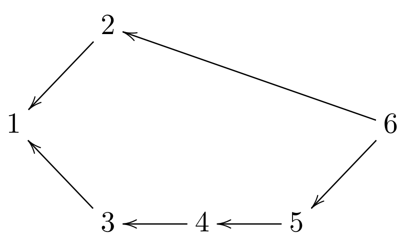
The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{2,3})}$

$R^1_\infty(1)$: –

$R^2_\infty(1) : \big(I(0,3), P(0,1)\big),\ \big(I(0,4), P(0,3)\big),\ \big(I(0,5), P(0,4)\big)$



A.6 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{2,4})$ – $\delta = \begin{smallmatrix} & 1 & & & \\ 1 & & & & 1 \\ & 1 & 1 & 1 & \end{smallmatrix}$



$$C_{\Delta(\widetilde{\mathbb{A}}_{2,4})} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Phi_{\Delta(\widetilde{\mathbb{A}}_{2,4})} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1)$: –

$P(1,1) : \big(R_0^3(1), P(0,2)\big),\ \big(R_\infty^1(1), P(0,3)\big)$

$P(2,1) : \big(R_0^2(3), P(0,5)\big),\ \big(R_0^3(2), P(0,6)\big),\ \big(R_0^4(1), P(1,2)\big),\ \big(R_\infty^2(1), P(1,3)\big),\ \big(I(1,6), 2P(0,4)\big)$

$P(n,1) : \big(R_0^{(n-1) \bmod 4+1}(3), P(n-2,5)\big),\ \big(R_0^{n \bmod 4+1}(2), P(n-2,6)\big),\ \big(R_0^{(n+1) \bmod 4+1}(1), P(n-1,2)\big)$
$\big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,3)\big),\ \big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$P(0,2) : \big(R_\infty^1(1), P(0,1)\big)$

$P(1,2) : \big(R_0^1(3), P(0,4)\big),\ \big(R_0^2(2), P(0,5)\big),\ \big(R_0^3(1), P(0,6)\big),\ \big(R_\infty^2(1), P(1,1)\big),\ \big(I(1,5), 2P(0,3)\big)$

$P(n,2) : \big(R_0^{(n-1) \bmod 4+1}(3), P(n-1,4)\big),\ \big(R_0^{n \bmod 4+1}(2), P(n-1,5)\big),\ \big(R_0^{(n+1) \bmod 4+1}(1), P(n-1,6)\big)$
$\big(R_\infty^{n \bmod 2+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n > 1$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$P(0,3) : \big(R_0^3(1), P(0,1)\big)$

$P(1,3) : \big(R_0^3(2), P(0,2)\big),\ \big(R_\infty^1(1), P(0,4)\big),\ \big(R_0^4(1), P(1,1)\big)$

$P(2,3) : \big(R_0^3(3), P(0,6)\big),\ \big(R_0^4(2), P(1,2)\big),\ \big(R_\infty^2(1), P(1,4)\big),\ \big(R_0^1(1), P(2,1)\big),\ \big(I(0,2), 2P(0,5)\big)$

$P(n,3) : \big(R_0^{n \bmod 4+1}(3), P(n-2,6)\big),\ \big(R_0^{(n+1) \bmod 4+1}(2), P(n-1,2)\big),\ \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,4)\big)$
$\big(R_0^{(n-2) \bmod 4+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4) = -1$, for $n \geq 0$.

$$P(0,4) : \big(R_0^3(2), P(0,1)\big),\ \big(R_0^4(1), P(0,3)\big)$$
$$P(1,4) : \big(R_0^3(3), P(0,2)\big),\ \big(R_\infty^1(1), P(0,5)\big),\ \big(R_0^4(2), P(1,1)\big),\ \big(R_0^1(1), P(1,3)\big)$$
$$P(2,4) : \big(R_0^4(3), P(1,2)\big),\ \big(R_\infty^2(1), P(1,5)\big),\ \big(R_0^1(2), P(2,1)\big),\ \big(R_0^2(1), P(2,3)\big),\ \big(2I(1,4), 3P(0,1)\big)$$
$$P(n,4) : \big(R_0^{(n+1) \bmod 4+1}(3), P(n-1,2)\big),\ \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,5)\big),\ \big(R_0^{(n-2) \bmod 4+1}(2), P(n,1)\big)$$
$$\big(R_0^{(n-1) \bmod 4+1}(1), P(n,3)\big),\ \big(uI, (u+1)P\big),\ n > 2$$

Modules of the form $P(n,5)$

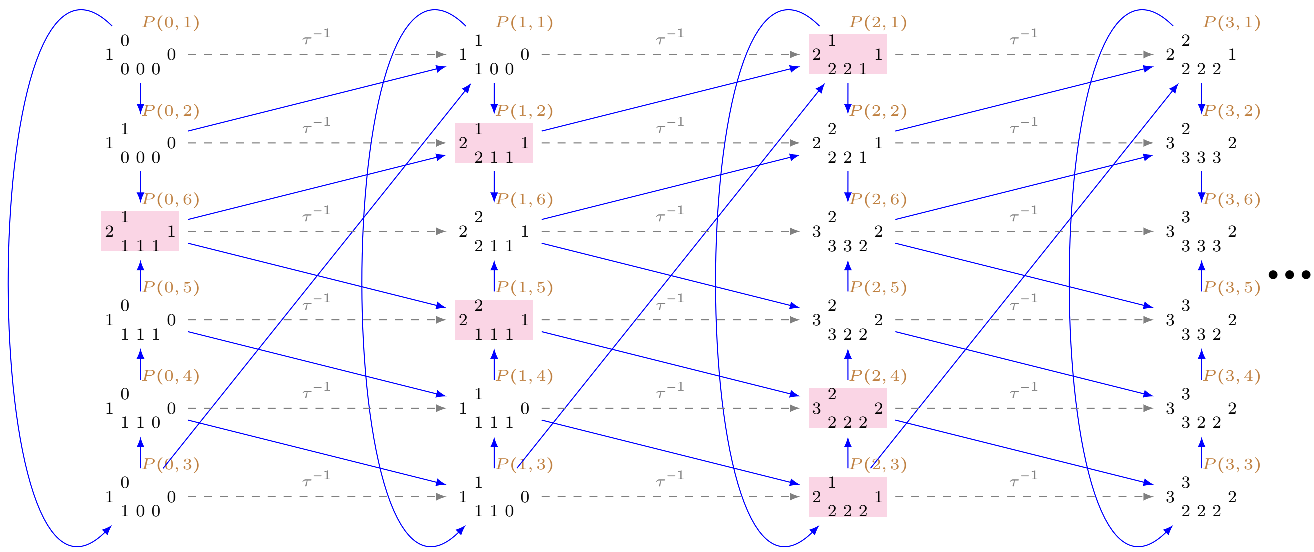
Defect: $\partial P(n,5) = -1$, for $n \geq 0$.

$$P(0,5) : \big(R_0^3(3), P(0,1)\big),\ \big(R_0^4(2), P(0,3)\big),\ \big(R_0^1(1), P(0,4)\big)$$
$$P(1,5) : \big(R_\infty^1(1), P(0,6)\big),\ \big(R_0^4(3), P(1,1)\big),\ \big(R_0^1(2), P(1,3)\big),\ \big(R_0^2(1), P(1,4)\big),\ \big(I(0,3), 2P(0,2)\big)$$
$$P(n,5) : \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-1,6)\big),\ \big(R_0^{(n+2) \bmod 4+1}(3), P(n,1)\big),\ \big(R_0^{(n-1) \bmod 4+1}(2), P(n,3)\big)$$
$$\big(R_0^{n \bmod 4+1}(1), P(n,4)\big),\ \big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,6)$

Defect: $\partial P(n,6) = -1$, for $n \geq 0$.

$$P(0,6) : \big(R_\infty^2(1), P(0,2)\big),\ \big(R_0^4(3), P(0,3)\big),\ \big(R_0^1(2), P(0,4)\big),\ \big(R_0^2(1), P(0,5)\big),\ \big(I(1,4), 2P(0,1)\big)$$
$$P(n,6) : \big(R_\infty^{(n+1) \bmod 2+1}(1), P(n,2)\big),\ \big(R_0^{(n+3) \bmod 4+1}(3), P(n,3)\big),\ \big(R_0^{n \bmod 4+1}(2), P(n,4)\big)$$
$$\big(R_0^{(n+1) \bmod 4+1}(1), P(n,5)\big),\ \big(uI, (u+1)P\big),\ n > 0$$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$$I(0,1) : \big(I(0,2), R^2_\infty(1)\big),\ \big(I(0,3), R^2_0(1)\big),\ \big(I(0,4), R^2_0(2)\big),\ \big(I(0,5), R^2_0(3)\big),\ \big(2I(0,6), P(1,4)\big)$$

$$I(n,1) : \big(I(n,2), R^{(-n+1) \bmod 2+1}_\infty(1)\big),\ \big(I(n,3), R^{(-n+1) \bmod 4+1}_0(1)\big),\ \big(I(n,4), R^{(-n+1) \bmod 4+1}_0(2)\big)$$
$$\big(I(n,5), R^{(-n+1) \bmod 4+1}_0(3)\big),\ \big((v+1)I, vP\big),\ n > 0$$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$$I(0,2) : \big(I(0,6), R^1_\infty(1)\big)$$

$$I(1,2) : \big(I(0,1), R^1_0(1)\big),\ \big(I(0,3), R^1_0(2)\big),\ \big(I(0,4), R^1_0(3)\big),\ \big(I(1,6), R^2_\infty(1)\big),\ \big(2I(0,5), P(1,3)\big)$$

$$I(n,2) : \big(I(n-1,1), R^{(-n+1) \bmod 4+1}_0(1)\big),\ \big(I(n-1,3), R^{(-n+1) \bmod 4+1}_0(2)\big),\ \big(I(n-1,4), R^{(-n+1) \bmod 4+1}_0(3)\big)$$
$$\big(I(n,6), R^{(-n+2) \bmod 2+1}_\infty(1)\big),\ \big((v+1)I, vP\big),\ n > 1$$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$I(0,3) : \big(I(0,4), R_0^3(1)\big),\ \big(I(0,5), R_0^3(2)\big),\ \big(I(0,6), R_0^3(3)\big)$
$I(1,3) : \big(I(0,1), R_\infty^1(1)\big),\ \big(I(1,4), R_0^2(1)\big),\ \big(I(1,5), R_0^2(2)\big),\ \big(I(1,6), R_0^2(3)\big),\ \big(2I(0,2), P(0,5)\big)$
$I(n,3) : \big(I(n-1,1), R_\infty^{(-n+1) \bmod 2+1}(1)\big),\ \big(I(n,4), R_0^{(-n+2) \bmod 4+1}(1)\big),\ \big(I(n,5), R_0^{(-n+2) \bmod 4+1}(2)\big)$
$\big(I(n,6), R_0^{(-n+2) \bmod 4+1}(3)\big),\ \big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$I(0,4) : \big(I(0,5), R_0^4(1)\big),\ \big(I(0,6), R_0^4(2)\big)$
$I(1,4) : \big(I(0,2), R_0^3(3)\big),\ \big(I(0,3), R_\infty^1(1)\big),\ \big(I(1,5), R_0^3(1)\big),\ \big(I(1,6), R_0^3(2)\big)$
$I(2,4) : \big(I(1,2), R_0^2(3)\big),\ \big(I(1,3), R_\infty^2(1)\big),\ \big(I(2,5), R_0^2(1)\big),\ \big(I(2,6), R_0^2(2)\big),\ \big(3I(0,6), 2P(1,4)\big)$
$I(n,4) : \big(I(n-1,2), R_0^{(-n+3) \bmod 4+1}(3)\big),\ \big(I(n-1,3), R_\infty^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n,5), R_0^{(-n+3) \bmod 4+1}(1)\big)$
$\big(I(n,6), R_0^{(-n+3) \bmod 4+1}(2)\big),\ \big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,5)$

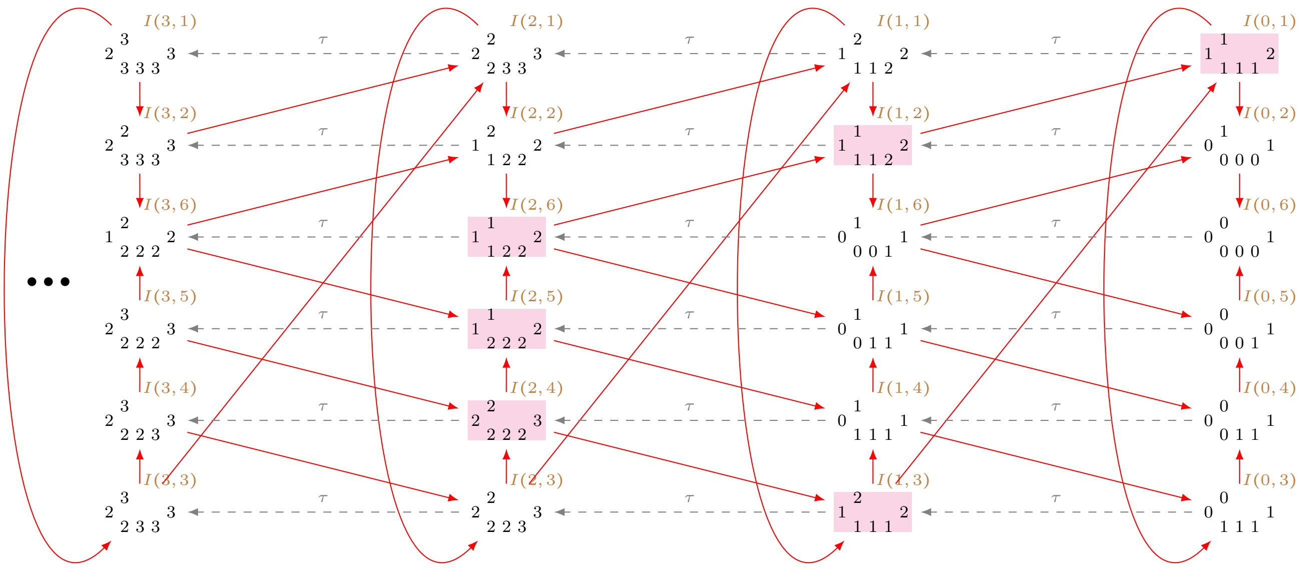
Defect: $\partial I(n,5) = 1$, for $n \geq 0$.

$I(0,5) : \big(I(0,6), R_0^1(1)\big)$
$I(1,5) : \big(I(0,2), R_0^4(2)\big),\ \big(I(0,4), R_\infty^1(1)\big),\ \big(I(1,6), R_0^4(1)\big)$
$I(2,5) : \big(I(0,1), R_0^3(3)\big),\ \big(I(1,2), R_0^3(2)\big),\ \big(I(1,4), R_\infty^2(1)\big),\ \big(I(2,6), R_0^3(1)\big),\ \big(2I(0,3), P(0,2)\big)$
$I(n,5) : \big(I(n-2,1), R_0^{(-n+4) \bmod 4+1}(3)\big),\ \big(I(n-1,2), R_0^{(-n+4) \bmod 4+1}(2)\big),\ \big(I(n-1,4), R_\infty^{(-n+3) \bmod 2+1}(1)\big)$
$\big(I(n,6), R_0^{(-n+4) \bmod 4+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,6)$

Defect: $\partial I(n,6) = 1$, for $n \geq 0$.

$I(0,6) : -$
$I(1,6) : \big(I(0,2), R_0^1(1)\big),\ \big(I(0,5), R_\infty^1(1)\big)$
$I(2,6) : \big(I(0,1), R_0^4(2)\big),\ \big(I(0,3), R_0^4(3)\big),\ \big(I(1,2), R_0^4(1)\big),\ \big(I(1,5), R_\infty^2(1)\big),\ \big(2I(0,4), P(1,1)\big)$
$I(n,6) : \big(I(n-2,1), R_0^{(-n+5) \bmod 4+1}(2)\big),\ \big(I(n-2,3), R_0^{(-n+5) \bmod 4+1}(3)\big),\ \big(I(n-1,2), R_0^{(-n+5) \bmod 4+1}(1)\big)$
$\big(I(n-1,5), R_\infty^{(-n+3) \bmod 2+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 2$



Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{2,4})}$

$R_0^1(1): -$

$R_0^1(2): \big(R_0^2(1), R_0^1(1)\big),\ \big(I(1,6), P(0,1)\big),\ \big(I(0,5), P(0,2)\big)$

$R_0^2(1): \big(I(0,2), P(0,1)\big),\ \big(I(0,6), P(0,2)\big)$

$R_0^2(2): \big(R_0^3(1), R_0^2(1)\big),\ \big(I(0,2), P(0,3)\big),\ \big(I(0,6), P(1,1)\big)$

$R_0^3(1): -$

$R_0^3(2): \big(R_0^4(1), R_0^3(1)\big)$

$R_0^4(1): -$

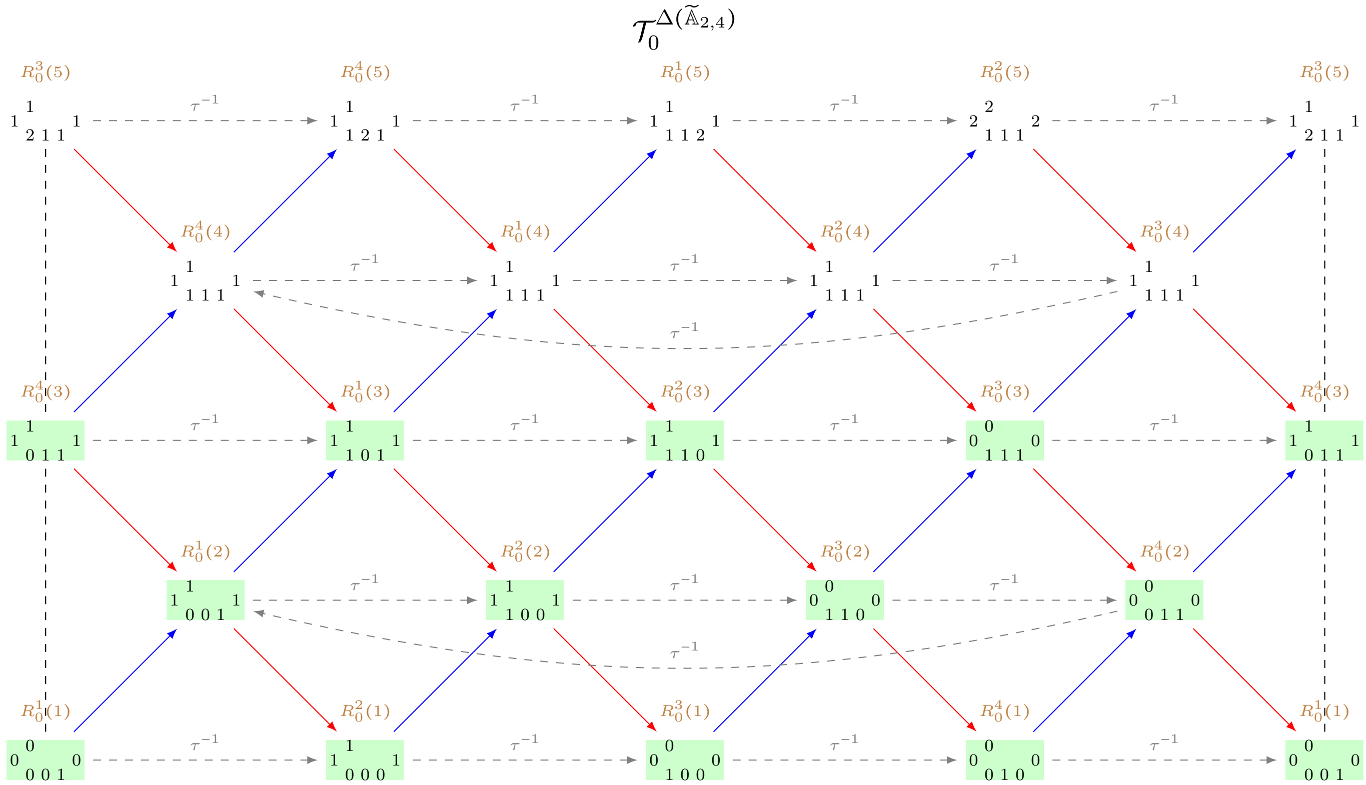
$R_0^4(2): \big(R_0^1(1), R_0^4(1)\big)$

$R_0^4(3): \big(R_0^1(2), R_0^4(1)\big),\ \big(R_0^2(1), R_0^4(2)\big),\ \big(I(1,5), P(0,1)\big),\ \big(I(0,4), P(0,2)\big)$

$R_0^1(3): \big(R_0^2(2), R_0^1(1)\big),\ \big(R_0^3(1), R_0^1(2)\big),\ \big(I(1,6), P(0,3)\big),\ \big(I(0,5), P(1,1)\big)$

$R_0^2(3): \big(R_0^3(2), R_0^2(1)\big),\ \big(R_0^4(1), R_0^2(2)\big),\ \big(I(0,2), P(0,4)\big),\ \big(I(0,6), P(1,3)\big)$

$R_0^3(3) : \big(R_0^4(2), R_0^3(1)\big),\ \big(R_0^1(1), R_0^3(2)\big)$

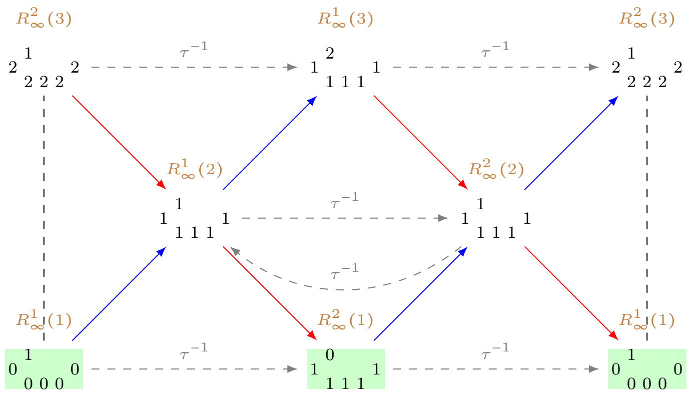


The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{2,4})}$

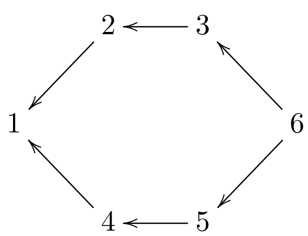
$R_\infty^1(1) : -$

$R_\infty^2(1) : \big(I(0,3), P(0,1)\big),\ \big(I(0,4), P(0,3)\big),\ \big(I(0,5), P(0,4)\big),\ \big(I(0,6), P(0,5)\big)$

$\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{2,4})}$



A.7 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{3,3})$ – $\delta = {}_1{}^{1\,1}_{1\,1}{}_1$



$$C_{\Delta(\widetilde{\mathbb{A}}_{3,3})} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{A}}_{3,3})} = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1)$: –
$P(1,1) : \big(R^3_0(1), P(0,2)\big),\ \big(R^3_\infty(1), P(0,4)\big)$
$P(2,1) : \big(R^3_0(2), P(0,3)\big),\ \big(R^3_\infty(2), P(0,5)\big),\ \big(R^1_0(1), P(1,2)\big),\ \big(R^1_\infty(1), P(1,4)\big)$
$P(3,1) : \big(R^1_0(2), P(1,3)\big),\ \big(R^1_\infty(2), P(1,5)\big),\ \big(R^2_0(1), P(2,2)\big),\ \big(R^2_\infty(1), P(2,4)\big),\ \big(2I(2,6), 3P(0,1)\big)$
$P(n,1) : \big(R_0^{(n-3) \bmod 3+1}(2), P(n-2,3)\big),\ \big(R_\infty^{(n-3) \bmod 3+1}(2), P(n-2,5)\big),\ \big(R_0^{(n-2) \bmod 3+1}(1), P(n-1,2)\big)$
$\big(R_\infty^{(n-2) \bmod 3+1}(1), P(n-1,4)\big),\ \big(uI, (u+1)P\big),\ n > 3$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$P(0,2) : \big(R^3_\infty(1), P(0,1)\big)$
$P(1,2) : \big(R^3_0(1), P(0,3)\big),\ \big(R^3_\infty(2), P(0,4)\big),\ \big(R^1_\infty(1), P(1,1)\big)$

$P(2,2) : \big(R_0^3(2), P(0,6)\big),\ \big(R_0^1(1), P(1,3)\big),\ \big(R_\infty^1(2), P(1,4)\big),\ \big(R_\infty^2(1), P(2,1)\big),\ \big(I(0,2), 2P(0,5)\big)$
$P(n,2) : \big(R_0^{n \bmod 3+1}(2), P(n-2,6)\big),\ \big(R_0^{(n-2) \bmod 3+1}(1), P(n-1,3)\big),\ \big(R_\infty^{(n-2) \bmod 3+1}(2), P(n-1,4)\big)$
$\big(R_\infty^{(n-1) \bmod 3+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$P(0,3) : \big(R_\infty^3(2), P(0,1)\big),\ \big(R_\infty^1(1), P(0,2)\big)$
$P(1,3) : \big(R_0^2(2), P(0,5)\big),\ \big(R_0^3(1), P(0,6)\big),\ \big(R_\infty^1(2), P(1,1)\big),\ \big(R_\infty^2(1), P(1,2)\big),\ \big(I(1,3), 2P(0,4)\big)$
$P(n,3) : \big(R_0^{n \bmod 3+1}(2), P(n-1,5)\big),\ \big(R_0^{(n+1) \bmod 3+1}(1), P(n-1,6)\big),\ \big(R_\infty^{(n-1) \bmod 3+1}(2), P(n,1)\big)$
$\big(R_\infty^{n \bmod 3+1}(1), P(n,2)\big),\ \big(uI, (u+1)P\big),\ n > 1$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4) = -1$, for $n \geq 0$.

$P(0,4) : \big(R_0^3(1), P(0,1)\big)$
$P(1,4) : \big(R_0^3(2), P(0,2)\big),\ \big(R_\infty^3(1), P(0,5)\big),\ \big(R_0^1(1), P(1,1)\big)$
$P(2,4) : \big(R_\infty^3(2), P(0,6)\big),\ \big(R_0^1(2), P(1,2)\big),\ \big(R_\infty^1(1), P(1,5)\big),\ \big(R_0^2(1), P(2,1)\big),\ \big(I(0,4), 2P(0,3)\big)$
$P(n,4) : \big(R_\infty^{n \bmod 3+1}(2), P(n-2,6)\big),\ \big(R_0^{(n-2) \bmod 3+1}(2), P(n-1,2)\big),\ \big(R_\infty^{(n-2) \bmod 3+1}(1), P(n-1,5)\big)$
$\big(R_0^{(n-1) \bmod 3+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -1$, for $n \geq 0$.

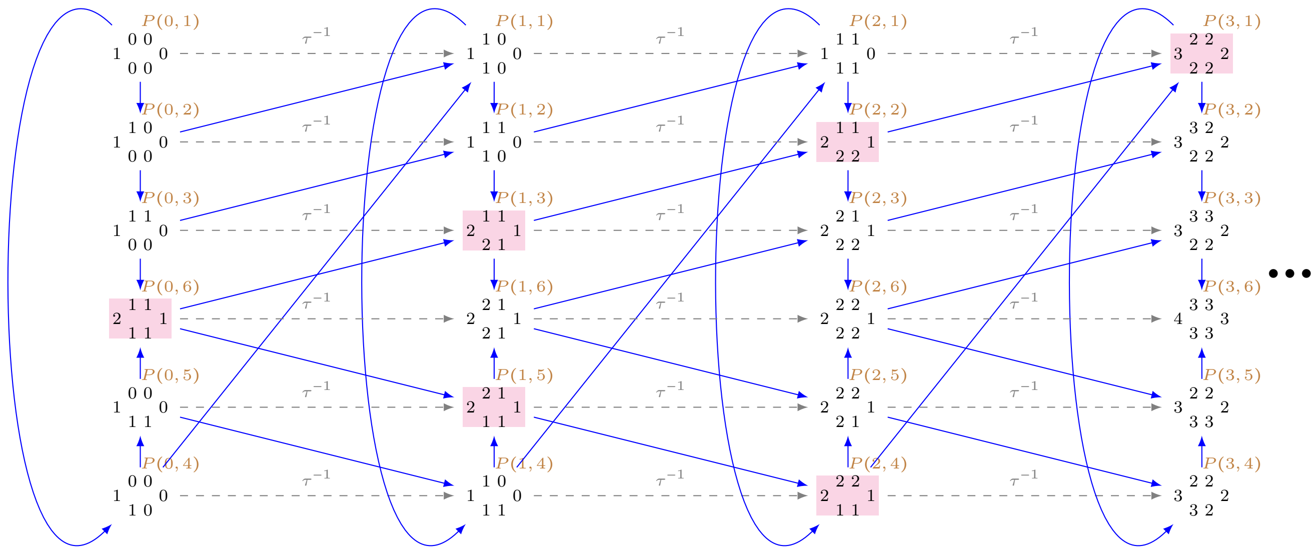
$P(0,5) : \big(R_0^3(2), P(0,1)\big),\ \big(R_0^1(1), P(0,4)\big)$
$P(1,5) : \big(R_\infty^2(2), P(0,3)\big),\ \big(R_\infty^3(1), P(0,6)\big),\ \big(R_0^1(2), P(1,1)\big),\ \big(R_0^2(1), P(1,4)\big),\ \big(I(1,5), 2P(0,2)\big)$
$P(n,5) : \big(R_\infty^{n \bmod 3+1}(2), P(n-1,3)\big),\ \big(R_\infty^{(n+1) \bmod 3+1}(1), P(n-1,6)\big),\ \big(R_0^{(n-1) \bmod 3+1}(2), P(n,1)\big)$
$\big(R_0^{n \bmod 3+1}(1), P(n,4)\big),\ \big(uI, (u+1)P\big),\ n > 1$

Modules of the form $P(n,6)$

Defect: $\partial P(n,6) = -1$, for $n \geq 0$.

$P(0,6) : \big(R_\infty^1(2), P(0,2)\big),\ \big(R_\infty^2(1), P(0,3)\big),\ \big(R_0^1(2), P(0,4)\big),\ \big(R_0^2(1), P(0,5)\big),\ \big(I(2,6), 2P(0,1)\big)$
$P(n,6) : \big(R_\infty^{n \bmod 3+1}(2), P(n,2)\big),\ \big(R_\infty^{(n+1) \bmod 3+1}(1), P(n,3)\big),\ \big(R_0^{n \bmod 3+1}(2), P(n,4)\big)$

$\big(R_0^{(n+1) \text{ mod } 3+1}(1), P(n,5)\big),\ \big(uI, (u+1)P\big),\ n > 0$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$$I(0,1) : \big(I(0,2), R_\infty^2(1)\big),\ \big(I(0,3), R_\infty^2(2)\big),\ \big(I(0,4), R_0^2(1)\big),\ \big(I(0,5), R_0^2(2)\big),\ \big(2I(0,6), P(2,1)\big)$$

$$I(n,1) : \big(I(n,2), R_\infty^{(-n+1) \text{ mod } 3+1}(1)\big),\ \big(I(n,3), R_\infty^{(-n+1) \text{ mod } 3+1}(2)\big),\ \big(I(n,4), R_0^{(-n+1) \text{ mod } 3+1}(1)\big)$$
$$\big(I(n,5), R_0^{(-n+1) \text{ mod } 3+1}(2)\big),\ \big((v+1)I, vP\big),\ n > 0$$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$$I(0,2) : \big(I(0,3), R_\infty^3(1)\big),\ \big(I(0,6), R_\infty^3(2)\big)$$
$$I(1,2) : \big(I(0,1), R_0^1(1)\big),\ \big(I(0,4), R_0^1(2)\big),\ \big(I(1,3), R_\infty^2(1)\big),\ \big(I(1,6), R_\infty^2(2)\big),\ \big(2I(0,5), P(1,2)\big)$$

$I(n,2):\big(I(n-1,1),R_0^{(-n+1)\bmod 3+1}(1)\big),\ \big(I(n-1,4),R_0^{(-n+1)\bmod 3+1}(2)\big),\ \big(I(n,3),R_\infty^{(-n+2)\bmod 3+1}(1)\big)$
$\big(I(n,6),R_\infty^{(-n+2)\bmod 3+1}(2)\big),\ \big((v+1)I,vP\big),\ n>1$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3)=1$, for $n\geq 0$.

$I(0,3):\big(I(0,6),R_\infty^1(1)\big)$
$I(1,3):\big(I(0,2),R_0^1(1)\big),\ \big(I(0,5),R_\infty^3(2)\big),\ \big(I(1,6),R_\infty^3(1)\big)$
$I(2,3):\big(I(0,1),R_0^3(2)\big),\ \big(I(1,2),R_0^3(1)\big),\ \big(I(1,5),R_\infty^2(2)\big),\ \big(I(2,6),R_\infty^2(1)\big),\ \big(2I(0,4),P(0,3)\big)$
$I(n,3):\big(I(n-2,1),R_0^{(-n+4)\bmod 3+1}(2)\big),\ \big(I(n-1,2),R_0^{(-n+4)\bmod 3+1}(1)\big),\ \big(I(n-1,5),R_\infty^{(-n+3)\bmod 3+1}(2)\big)$
$\big(I(n,6),R_\infty^{(-n+3)\bmod 3+1}(1)\big),\ \big((v+1)I,vP\big),\ n>2$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4)=1$, for $n\geq 0$.

$I(0,4):\big(I(0,5),R_0^3(1)\big),\ \big(I(0,6),R_0^3(2)\big)$
$I(1,4):\big(I(0,1),R_\infty^1(1)\big),\ \big(I(0,2),R_\infty^1(2)\big),\ \big(I(1,5),R_0^2(1)\big),\ \big(I(1,6),R_0^2(2)\big),\ \big(2I(0,3),P(1,4)\big)$
$I(n,4):\big(I(n-1,1),R_\infty^{(-n+1)\bmod 3+1}(1)\big),\ \big(I(n-1,2),R_\infty^{(-n+1)\bmod 3+1}(2)\big),\ \big(I(n,5),R_0^{(-n+2)\bmod 3+1}(1)\big)$
$\big(I(n,6),R_0^{(-n+2)\bmod 3+1}(2)\big),\ \big((v+1)I,vP\big),\ n>1$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5)=1$, for $n\geq 0$.

$I(0,5):\big(I(0,6),R_0^1(1)\big)$
$I(1,5):\big(I(0,3),R_0^3(2)\big),\ \big(I(0,4),R_\infty^1(1)\big),\ \big(I(1,6),R_0^3(1)\big)$
$I(2,5):\big(I(0,1),R_\infty^3(2)\big),\ \big(I(1,3),R_0^2(2)\big),\ \big(I(1,4),R_\infty^3(1)\big),\ \big(I(2,6),R_0^2(1)\big),\ \big(2I(0,2),P(0,5)\big)$
$I(n,5):\big(I(n-2,1),R_\infty^{(-n+4)\bmod 3+1}(2)\big),\ \big(I(n-1,3),R_0^{(-n+3)\bmod 3+1}(2)\big),\ \big(I(n-1,4),R_\infty^{(-n+4)\bmod 3+1}(1)\big)$
$\big(I(n,6),R_0^{(-n+3)\bmod 3+1}(1)\big),\ \big((v+1)I,vP\big),\ n>2$

Modules of the form $I(n,6)$

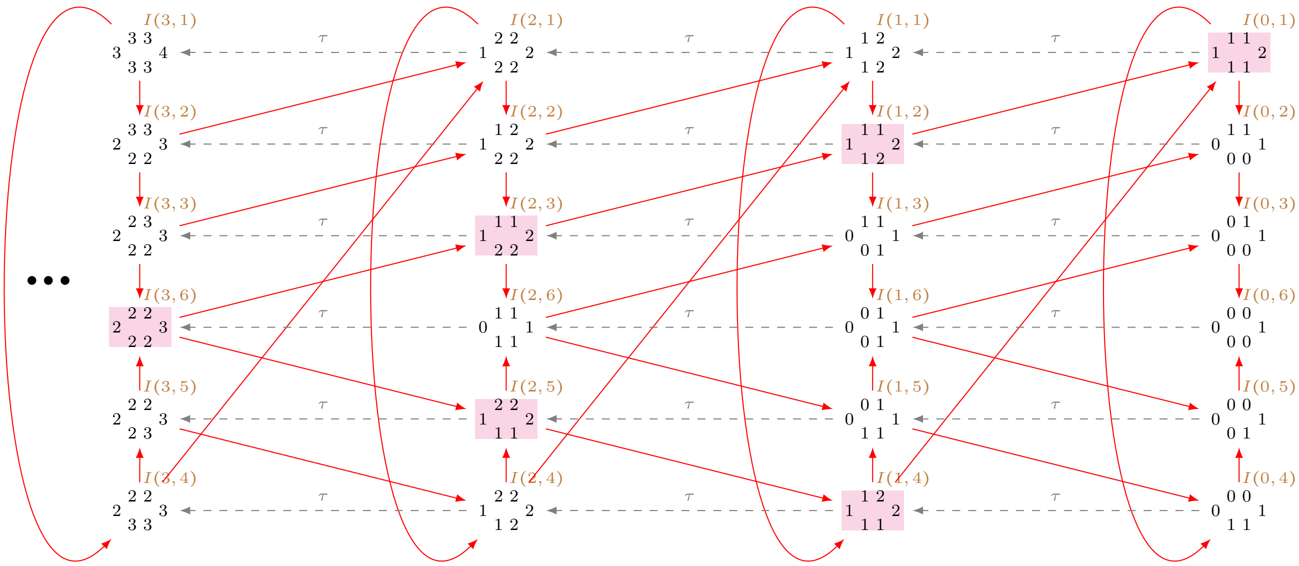
Defect: $\partial I(n,6)=1$, for $n\geq 0$.

$I(0,6):-$
$I(1,6):\big(I(0,3),R_0^1(1)\big),\ \big(I(0,5),R_\infty^1(1)\big)$

$I(2,6) : \big(I(0,2), R_0^3(2)\big),\ \big(I(0,4), R_\infty^3(2)\big),\ \big(I(1,3), R_0^3(1)\big),\ \big(I(1,5), R_\infty^3(1)\big)$

$I(3,6) : \big(I(1,2), R_0^2(2)\big),\ \big(I(1,4), R_\infty^2(2)\big),\ \big(I(2,3), R_0^2(1)\big),\ \big(I(2,5), R_\infty^2(1)\big),\ \big(3I(0,6), 2P(2,1)\big)$

$I(n,6) : \big(I(n-2,2), R_0^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-2,4), R_\infty^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,3), R_0^{(-n+4) \bmod 3+1}(1)\big)$
$\big(I(n-1,5), R_\infty^{(-n+4) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 3$



Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{3,3})}$

$R_0^1(1) : -$

$R_0^1(2) : \big(R_0^2(1), R_0^1(1)\big),\ \big(I(1,3), P(0,1)\big),\ \big(I(1,6), P(0,2)\big),\ \big(I(0,5), P(0,3)\big)$

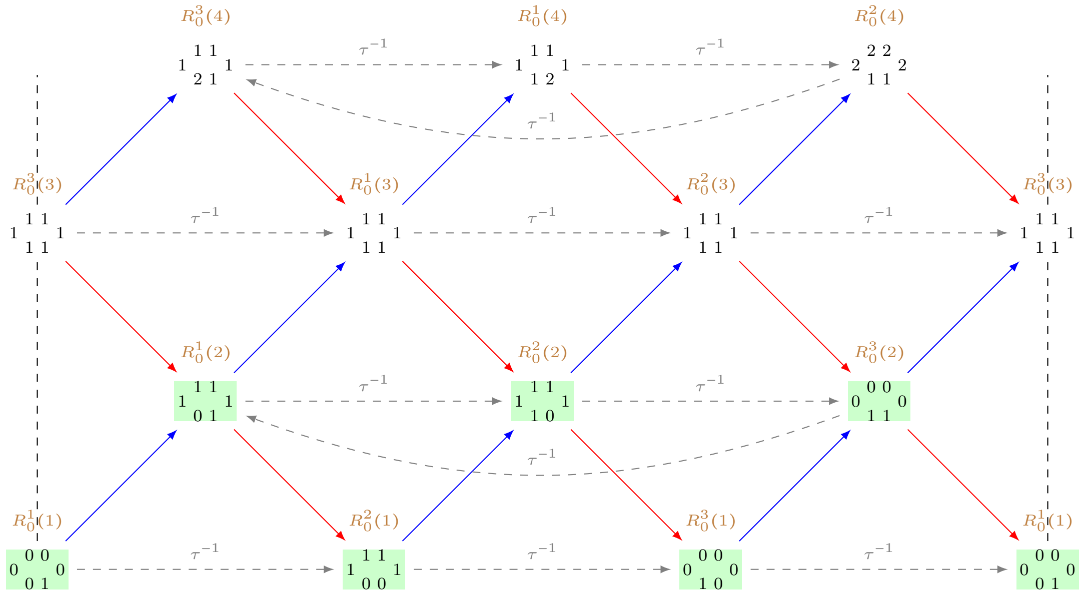
$R_0^2(1) : \big(I(0,2), P(0,1)\big),\ \big(I(0,3), P(0,2)\big),\ \big(I(0,6), P(0,3)\big)$

$R_0^2(2) : \big(R_0^3(1), R_0^2(1)\big),\ \big(I(0,2), P(0,4)\big),\ \big(I(0,3), P(1,1)\big),\ \big(I(0,6), P(1,2)\big)$

$R_0^3(1) : -$

$R_0^3(2) : \big(R_0^1(1), R_0^3(1)\big)$

$\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{3,3})}$



The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{3,3})}$

$R^1_\infty(1)$: –

$R^1_\infty(2) : \big(R^2_\infty(1), R^1_\infty(1)\big),\ \big(I(1,5), P(0,1)\big),\ \big(I(1,6), P(0,4)\big),\ \big(I(0,3), P(0,5)\big)$

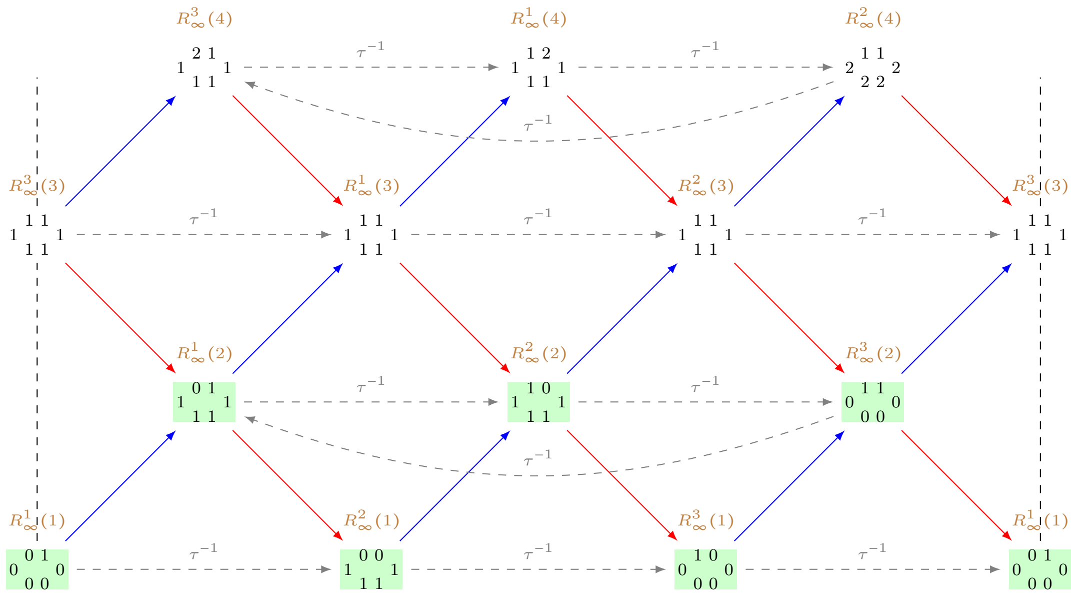
$R^2_\infty(1) : \big(I(0,4), P(0,1)\big),\ \big(I(0,5), P(0,4)\big),\ \big(I(0,6), P(0,5)\big)$

$R^2_\infty(2) : \big(R^3_\infty(1), R^2_\infty(1)\big),\ \big(I(0,4), P(0,2)\big),\ \big(I(0,5), P(1,1)\big),\ \big(I(0,6), P(1,4)\big)$

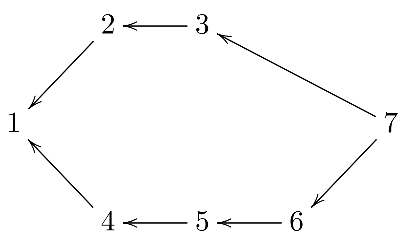
$R^3_\infty(1)$: –

$R^3_\infty(2) : \big(R^1_\infty(1), R^3_\infty(1)\big)$

$\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{3,3})}$



A.8 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{3,4})$ – $\delta = {}_1\,{}^{1\,1}_{1\,1\,1}\,{}_1$



$$C_{\Delta(\widetilde{\mathbb{A}}_{3,4})} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Phi_{\Delta(\widetilde{\mathbb{A}}_{3,4})} = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1)$: –
$P(1,1)$: $\big(R^3_0(1), P(0,2)\big)$, $\big(R^3_\infty(1), P(0,4)\big)$

$P(2,1):\big(R_0^3(2),P(0,3)\big),\ \big(R_\infty^3(2),P(0,5)\big),\ \big(R_0^4(1),P(1,2)\big),\ \big(R_\infty^1(1),P(1,4)\big)$
$P(3,1):\big(R_0^3(3),P(0,7)\big),\ \big(R_0^4(2),P(1,3)\big),\ \big(R_\infty^1(2),P(1,5)\big),\ \big(R_0^1(1),P(2,2)\big),\ \big(R_\infty^2(1),P(2,4)\big)$
$\big(I(0,2),2P(0,6)\big)$
$P(n,1):\big(R_0^{(n-1)\bmod 4+1}(3),P(n-3,7)\big),\ \big(R_0^{n\bmod 4+1}(2),P(n-2,3)\big),\ \big(R_\infty^{(n-3)\bmod 3+1}(2),P(n-2,5)\big)$
$\big(R_0^{(n-3)\bmod 4+1}(1),P(n-1,2)\big),\ \big(R_\infty^{(n-2)\bmod 3+1}(1),P(n-1,4)\big),\ \big(uI,(u+1)P\big),\ n>3$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2)=-1$, for $n\geq 0$.

$P(0,2):\big(R_\infty^3(1),P(0,1)\big)$
$P(1,2):\big(R_0^3(1),P(0,3)\big),\ \big(R_\infty^3(2),P(0,4)\big),\ \big(R_\infty^1(1),P(1,1)\big)$
$P(2,2):\big(R_0^2(3),P(0,6)\big),\ \big(R_0^3(2),P(0,7)\big),\ \big(R_0^4(1),P(1,3)\big),\ \big(R_\infty^1(2),P(1,4)\big),\ \big(R_\infty^2(1),P(2,1)\big)$
$\big(I(1,3),2P(0,5)\big)$
$P(n,2):\big(R_0^{(n-1)\bmod 4+1}(3),P(n-2,6)\big),\ \big(R_0^{n\bmod 4+1}(2),P(n-2,7)\big),\ \big(R_0^{(n+1)\bmod 4+1}(1),P(n-1,3)\big)$
$\big(R_\infty^{(n-2)\bmod 3+1}(2),P(n-1,4)\big),\ \big(R_\infty^{(n-1)\bmod 3+1}(1),P(n,1)\big),\ \big(uI,(u+1)P\big),\ n>2$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3)=-1$, for $n\geq 0$.

$P(0,3):\big(R_\infty^3(2),P(0,1)\big),\ \big(R_\infty^1(1),P(0,2)\big)$
$P(1,3):\big(R_0^1(3),P(0,5)\big),\ \big(R_0^2(2),P(0,6)\big),\ \big(R_0^3(1),P(0,7)\big),\ \big(R_\infty^1(2),P(1,1)\big),\ \big(R_\infty^2(1),P(1,2)\big)$
$\big(I(2,7),2P(0,4)\big)$
$P(n,3):\big(R_0^{(n-1)\bmod 4+1}(3),P(n-1,5)\big),\ \big(R_0^{n\bmod 4+1}(2),P(n-1,6)\big),\ \big(R_0^{(n+1)\bmod 4+1}(1),P(n-1,7)\big)$
$\big(R_\infty^{(n-1)\bmod 3+1}(2),P(n,1)\big),\ \big(R_\infty^{n\bmod 3+1}(1),P(n,2)\big),\ \big(uI,(u+1)P\big),\ n>1$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4)=-1$, for $n\geq 0$.

$P(0,4):\big(R_0^3(1),P(0,1)\big)$
$P(1,4):\big(R_0^3(2),P(0,2)\big),\ \big(R_\infty^3(1),P(0,5)\big),\ \big(R_0^4(1),P(1,1)\big)$
$P(2,4):\big(R_0^3(3),P(0,3)\big),\ \big(R_\infty^3(2),P(0,6)\big),\ \big(R_0^4(2),P(1,2)\big),\ \big(R_\infty^1(1),P(1,5)\big),\ \big(R_0^1(1),P(2,1)\big)$
$P(3,4):\big(R_0^4(3),P(1,3)\big),\ \big(R_\infty^1(2),P(1,6)\big),\ \big(R_0^1(2),P(2,2)\big),\ \big(R_\infty^2(1),P(2,5)\big),\ \big(R_0^2(1),P(3,1)\big)$
$\big(2I(2,6),3P(0,1)\big)$
$P(n,4):\big(R_0^{n\bmod 4+1}(3),P(n-2,3)\big),\ \big(R_\infty^{(n-3)\bmod 3+1}(2),P(n-2,6)\big),\ \big(R_0^{(n-3)\bmod 4+1}(2),P(n-1,2)\big)$

$\big(R_\infty^{(n-2) \bmod 3+1}(1), P(n-1,5)\big),\ \big(R_0^{(n-2) \bmod 4+1}(1), P(n,1)\big),\ \big(uI, (u+1)P\big),\ n>3$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -1$, for $n \geq 0$.

$P(0,5) : \big(R_0^3(2), P(0,1)\big),\ \big(R_0^4(1), P(0,4)\big)$
$P(1,5) : \big(R_0^3(3), P(0,2)\big),\ \big(R_\infty^3(1), P(0,6)\big),\ \big(R_0^4(2), P(1,1)\big),\ \big(R_0^1(1), P(1,4)\big)$
$P(2,5) : \big(R_\infty^3(2), P(0,7)\big),\ \big(R_0^4(3), P(1,2)\big),\ \big(R_\infty^1(1), P(1,6)\big),\ \big(R_0^1(2), P(2,1)\big),\ \big(R_0^2(1), P(2,4)\big)$
$\big(I(0,4), 2P(0,3)\big)$
$P(n,5) : \big(R_\infty^{n \bmod 3+1}(2), P(n-2,7)\big),\ \big(R_0^{(n+1) \bmod 4+1}(3), P(n-1,2)\big),\ \big(R_\infty^{(n-2) \bmod 3+1}(1), P(n-1,6)\big)$
$\big(R_0^{(n-2) \bmod 4+1}(2), P(n,1)\big),\ \big(R_0^{(n-1) \bmod 4+1}(1), P(n,4)\big),\ \big(uI, (u+1)P\big),\ n>2$

Modules of the form $P(n,6)$

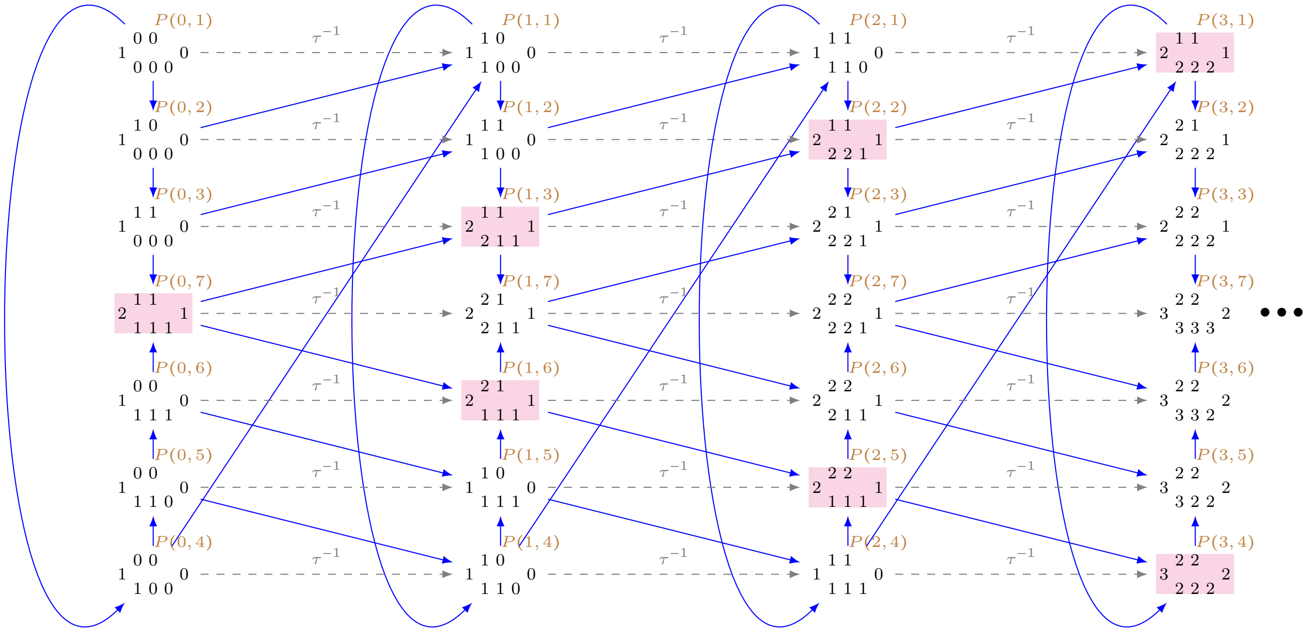
Defect: $\partial P(n,6) = -1$, for $n \geq 0$.

$P(0,6) : \big(R_0^3(3), P(0,1)\big),\ \big(R_0^4(2), P(0,4)\big),\ \big(R_0^1(1), P(0,5)\big)$
$P(1,6) : \big(R_\infty^2(2), P(0,3)\big),\ \big(R_\infty^3(1), P(0,7)\big),\ \big(R_0^4(3), P(1,1)\big),\ \big(R_0^1(2), P(1,4)\big),\ \big(R_0^2(1), P(1,5)\big)$
$\big(I(1,5), 2P(0,2)\big)$
$P(n,6) : \big(R_\infty^{n \bmod 3+1}(2), P(n-1,3)\big),\ \big(R_\infty^{(n+1) \bmod 3+1}(1), P(n-1,7)\big),\ \big(R_0^{(n+2) \bmod 4+1}(3), P(n,1)\big)$
$\big(R_0^{(n-1) \bmod 4+1}(2), P(n,4)\big),\ \big(R_0^{n \bmod 4+1}(1), P(n,5)\big),\ \big(uI, (u+1)P\big),\ n>1$

Modules of the form $P(n,7)$

Defect: $\partial P(n,7) = -1$, for $n \geq 0$.

$P(0,7) : \big(R_\infty^1(2), P(0,2)\big),\ \big(R_\infty^2(1), P(0,3)\big),\ \big(R_0^4(3), P(0,4)\big),\ \big(R_0^1(2), P(0,5)\big),\ \big(R_0^2(1), P(0,6)\big)$
$\big(I(2,6), 2P(0,1)\big)$
$P(n,7) : \big(R_\infty^{n \bmod 3+1}(2), P(n,2)\big),\ \big(R_\infty^{(n+1) \bmod 3+1}(1), P(n,3)\big),\ \big(R_0^{(n+3) \bmod 4+1}(3), P(n,4)\big)$
$\big(R_0^{n \bmod 4+1}(2), P(n,5)\big),\ \big(R_0^{(n+1) \bmod 4+1}(1), P(n,6)\big),\ \big(uI, (u+1)P\big),\ n>0$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$$I(0,1) : \big(I(0,2), R^2_\infty(1)\big),\ \big(I(0,3), R^2_\infty(2)\big),\ \big(I(0,4), R^2_0(1)\big),\ \big(I(0,5), R^2_0(2)\big),\ \big(I(0,6), R^2_0(3)\big)$$
$$\big(2I(0,7), P(2,4)\big)$$
$$I(n,1) : \big(I(n,2), R_\infty^{(-n+1) \bmod 3+1}(1)\big),\ \big(I(n,3), R_\infty^{(-n+1) \bmod 3+1}(2)\big),\ \big(I(n,4), R_0^{(-n+1) \bmod 4+1}(1)\big)$$
$$\big(I(n,5), R_0^{(-n+1) \bmod 4+1}(2)\big),\ \big(I(n,6), R_0^{(-n+1) \bmod 4+1}(3)\big),\ \big((v+1)I, vP\big),\ n > 0$$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$$I(0,2) : \big(I(0,3), R^3_\infty(1)\big),\ \big(I(0,7), R^3_\infty(2)\big)$$
$$I(1,2) : \big(I(0,1), R^1_0(1)\big),\ \big(I(0,4), R^1_0(2)\big),\ \big(I(0,5), R^1_0(3)\big),\ \big(I(1,3), R^2_\infty(1)\big),\ \big(I(1,7), R^2_\infty(2)\big)$$
$$\big(2I(0,6), P(2,1)\big)$$
$$I(n,2) : \big(I(n-1,1), R_0^{(-n+1) \bmod 4+1}(1)\big),\ \big(I(n-1,4), R_0^{(-n+1) \bmod 4+1}(2)\big),\ \big(I(n-1,5), R_0^{(-n+1) \bmod 4+1}(3)\big)$$

$\big(I(n,3), R_\infty^{(-n+2) \bmod 3+1}(1)\big),\ \big(I(n,7), R_\infty^{(-n+2) \bmod 3+1}(2)\big),\ \big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$I(0,3) : \big(I(0,7), R_\infty^1(1)\big)$
$I(1,3) : \big(I(0,2), R_0^1(1)\big),\ \big(I(0,6), R_\infty^3(2)\big),\ \big(I(1,7), R_\infty^3(1)\big)$
$I(2,3) : \big(I(0,1), R_0^4(2)\big),\ \big(I(0,4), R_0^4(3)\big),\ \big(I(1,2), R_0^4(1)\big),\ \big(I(1,6), R_\infty^2(2)\big),\ \big(I(2,7), R_\infty^2(1)\big)$
$\big(2I(0,5), P(1,2)\big)$
$I(n,3) : \big(I(n-2,1), R_0^{(-n+5) \bmod 4+1}(2)\big),\ \big(I(n-2,4), R_0^{(-n+5) \bmod 4+1}(3)\big),\ \big(I(n-1,2), R_0^{(-n+5) \bmod 4+1}(1)\big)$
$\big(I(n-1,6), R_\infty^{(-n+3) \bmod 3+1}(2)\big),\ \big(I(n,7), R_\infty^{(-n+3) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$I(0,4) : \big(I(0,5), R_0^3(1)\big),\ \big(I(0,6), R_0^3(2)\big),\ \big(I(0,7), R_0^3(3)\big)$
$I(1,4) : \big(I(0,1), R_\infty^1(1)\big),\ \big(I(0,2), R_\infty^1(2)\big),\ \big(I(1,5), R_0^2(1)\big),\ \big(I(1,6), R_0^2(2)\big),\ \big(I(1,7), R_0^2(3)\big)$
$\big(2I(0,3), P(1,5)\big)$
$I(n,4) : \big(I(n-1,1), R_\infty^{(-n+1) \bmod 3+1}(1)\big),\ \big(I(n-1,2), R_\infty^{(-n+1) \bmod 3+1}(2)\big),\ \big(I(n,5), R_0^{(-n+2) \bmod 4+1}(1)\big)$
$\big(I(n,6), R_0^{(-n+2) \bmod 4+1}(2)\big),\ \big(I(n,7), R_0^{(-n+2) \bmod 4+1}(3)\big),\ \big((v+1)I, vP\big),\ n > 1$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 1$, for $n \geq 0$.

$I(0,5) : \big(I(0,6), R_0^4(1)\big),\ \big(I(0,7), R_0^4(2)\big)$
$I(1,5) : \big(I(0,3), R_0^3(3)\big),\ \big(I(0,4), R_\infty^1(1)\big),\ \big(I(1,6), R_0^3(1)\big),\ \big(I(1,7), R_0^3(2)\big)$
$I(2,5) : \big(I(0,1), R_\infty^3(2)\big),\ \big(I(1,3), R_0^2(3)\big),\ \big(I(1,4), R_\infty^3(1)\big),\ \big(I(2,6), R_0^2(1)\big),\ \big(I(2,7), R_0^2(2)\big)$
$\big(2I(0,2), P(0,6)\big)$
$I(n,5) : \big(I(n-2,1), R_\infty^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,3), R_0^{(-n+3) \bmod 4+1}(3)\big),\ \big(I(n-1,4), R_\infty^{(-n+4) \bmod 3+1}(1)\big)$
$\big(I(n,6), R_0^{(-n+3) \bmod 4+1}(1)\big),\ \big(I(n,7), R_0^{(-n+3) \bmod 4+1}(2)\big),\ \big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,6)$

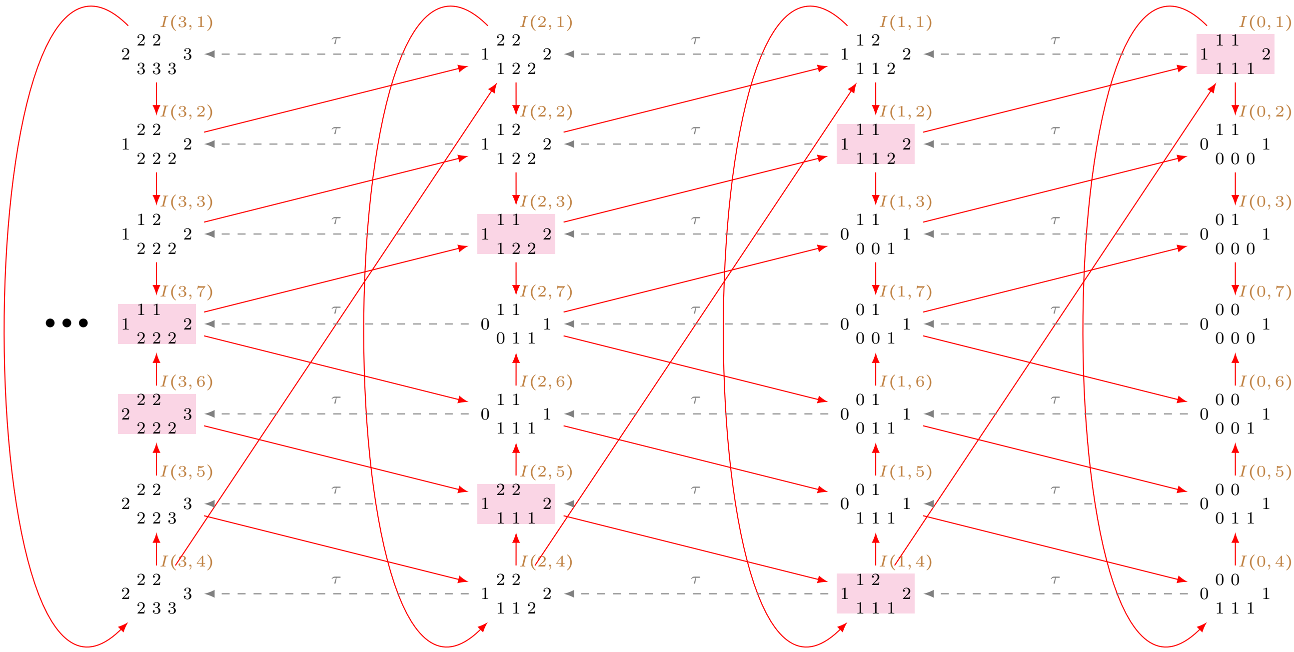
Defect: $\partial I(n,6) = 1$, for $n \geq 0$.

$I(0,6) : \big(I(0,7), R_0^1(1)\big)$
$I(1,6) : \big(I(0,3), R_0^4(2)\big),\ \big(I(0,5), R_\infty^1(1)\big),\ \big(I(1,7), R_0^4(1)\big)$
$I(2,6) : \big(I(0,2), R_0^3(3)\big),\ \big(I(0,4), R_\infty^3(2)\big),\ \big(I(1,3), R_0^3(2)\big),\ \big(I(1,5), R_\infty^3(1)\big),\ \big(I(2,7), R_0^3(1)\big)$
$I(3,6) : \big(I(1,2), R_0^2(3)\big),\ \big(I(1,4), R_\infty^2(2)\big),\ \big(I(2,3), R_0^2(2)\big),\ \big(I(2,5), R_\infty^2(1)\big),\ \big(I(3,7), R_0^2(1)\big)$
$\big(3I(0,7), 2P(2,4)\big)$
$I(n,6) : \big(I(n-2,2), R_0^{(-n+4) \bmod 4+1}(3)\big),\ \big(I(n-2,4), R_\infty^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,3), R_0^{(-n+4) \bmod 4+1}(2)\big)$
$\big(I(n-1,5), R_\infty^{(-n+4) \bmod 3+1}(1)\big),\ \big(I(n,7), R_0^{(-n+4) \bmod 4+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 3$

Modules of the form $I(n,7)$

Defect: $\partial I(n,7) = 1$, for $n \geq 0$.

$I(0,7) : -$
$I(1,7) : \big(I(0,3), R_0^1(1)\big),\ \big(I(0,6), R_\infty^1(1)\big)$
$I(2,7) : \big(I(0,2), R_0^4(2)\big),\ \big(I(0,5), R_\infty^3(2)\big),\ \big(I(1,3), R_0^4(1)\big),\ \big(I(1,6), R_\infty^3(1)\big)$
$I(3,7) : \big(I(0,1), R_0^3(3)\big),\ \big(I(1,2), R_0^3(2)\big),\ \big(I(1,5), R_\infty^2(2)\big),\ \big(I(2,3), R_0^3(1)\big),\ \big(I(2,6), R_\infty^2(1)\big)$
$\big(2I(0,4), P(0,3)\big)$
$I(n,7) : \big(I(n-3,1), R_0^{(-n+5) \bmod 4+1}(3)\big),\ \big(I(n-2,2), R_0^{(-n+5) \bmod 4+1}(2)\big),\ \big(I(n-2,5), R_\infty^{(-n+4) \bmod 3+1}(2)\big)$
$\big(I(n-1,3), R_0^{(-n+5) \bmod 4+1}(1)\big),\ \big(I(n-1,6), R_\infty^{(-n+4) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 3$



Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{3,4})}$

$R_0^1(1)$: –

$R_0^1(2) : \big(R_0^2(1), R_0^1(1)\big),\ \big(I(1,3), P(0,1)\big),\ \big(I(1,7), P(0,2)\big),\ \big(I(0,6), P(0,3)\big)$

$R_0^2(1) : \big(I(0,2), P(0,1)\big),\ \big(I(0,3), P(0,2)\big),\ \big(I(0,7), P(0,3)\big)$

$R_0^2(2) : \big(R_0^3(1), R_0^2(1)\big),\ \big(I(0,2), P(0,4)\big),\ \big(I(0,3), P(1,1)\big),\ \big(I(0,7), P(1,2)\big)$

$R_0^3(1)$: –

$R_0^3(2) : \big(R_0^4(1), R_0^3(1)\big)$

$R_0^4(1)$: –

$R_0^4(2) : \big(R_0^1(1), R_0^4(1)\big)$

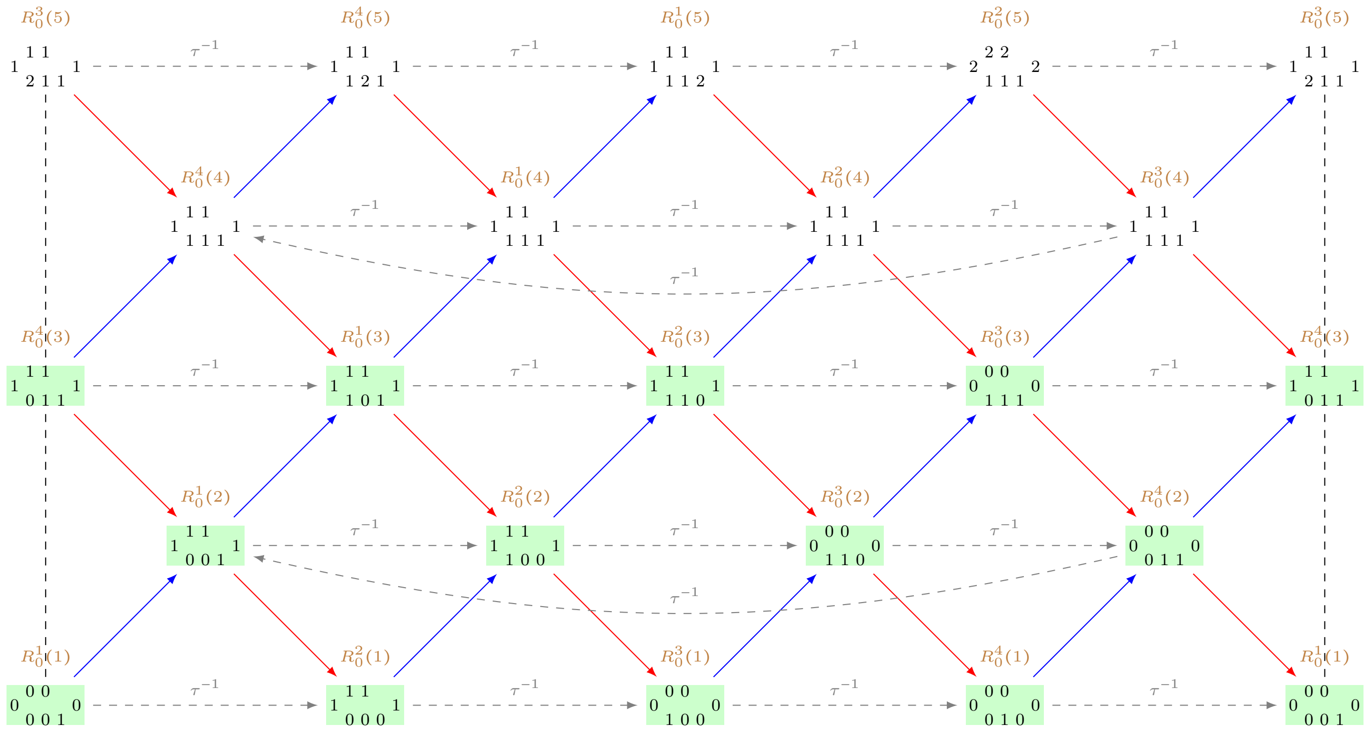
$R_0^4(3) : \big(R_0^1(2), R_0^4(1)\big),\ \big(R_0^2(1), R_0^4(2)\big),\ \big(I(2,7), P(0,1)\big),\ \big(I(1,6), P(0,2)\big),\ \big(I(0,5), P(0,3)\big)$

$R_0^1(3) : \big(R_0^2(2), R_0^1(1)\big),\ \big(R_0^3(1), R_0^1(2)\big),\ \big(I(1,3), P(0,4)\big),\ \big(I(1,7), P(1,1)\big),\ \big(I(0,6), P(1,2)\big)$

$R_0^2(3) : \big(R_0^3(2), R_0^2(1)\big),\ \big(R_0^4(1), R_0^2(2)\big),\ \big(I(0,2), P(0,5)\big),\ \big(I(0,3), P(1,4)\big),\ \big(I(0,7), P(2,1)\big)$

$R_0^3(3) : \big(R_0^4(2), R_0^3(1)\big),\ \big(R_0^1(1), R_0^3(2)\big)$

$\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{3,4})}$



The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{3,4})}$

$R^1_\infty(1)$: –

$R^1_\infty(2) : \big(R^2_\infty(1), R^1_\infty(1)\big),\ \big(I(1,5), P(0,1)\big),\ \big(I(1,6), P(0,4)\big),\ \big(I(1,7), P(0,5)\big),\ \big(I(0,3), P(0,6)\big)$

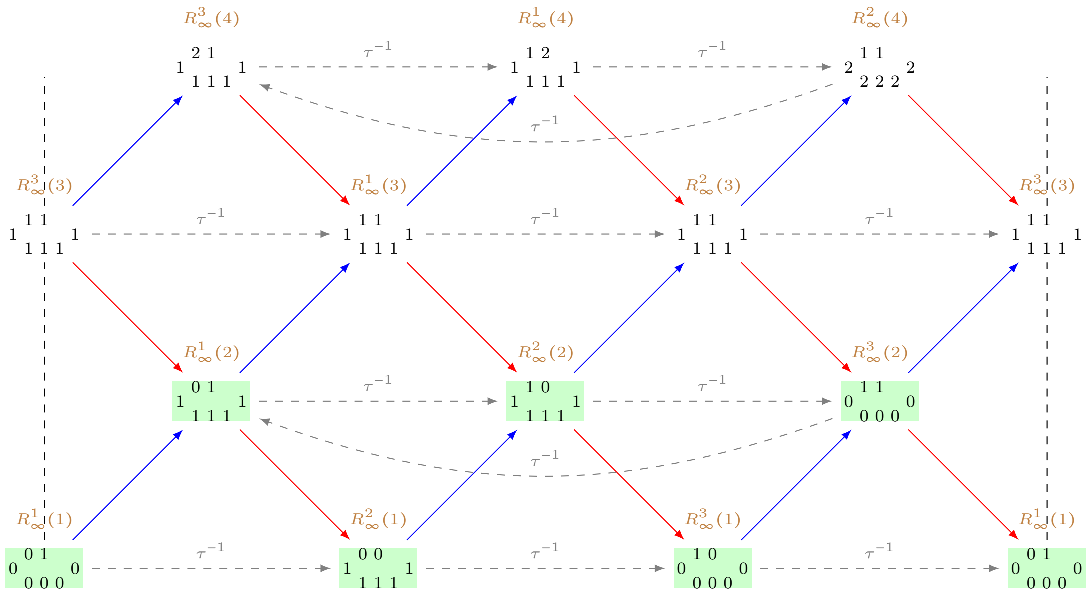
$R^2_\infty(1) : \big(I(0,4), P(0,1)\big),\ \big(I(0,5), P(0,4)\big),\ \big(I(0,6), P(0,5)\big),\ \big(I(0,7), P(0,6)\big)$

$R^2_\infty(2) : \big(R^3_\infty(1), R^2_\infty(1)\big),\ \big(I(0,4), P(0,2)\big),\ \big(I(0,5), P(1,1)\big),\ \big(I(0,6), P(1,4)\big),\ \big(I(0,7), P(1,5)\big)$

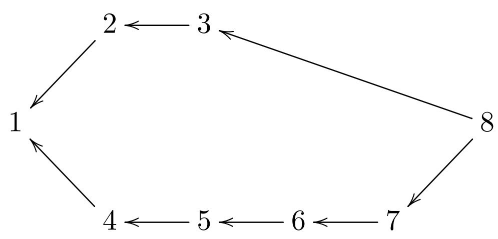
$R^3_\infty(1)$: –

$R^3_\infty(2) : \big(R^1_\infty(1), R^3_\infty(1)\big)$

$\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{3,4})}$



A.9 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{A}}_{3,5})$ – $\delta = 1 \begin{smallmatrix} 1\ 1 \\ 1\ 1\ 1\ 1 \end{smallmatrix} 1$



$$C_{\Delta(\widetilde{\mathbb{A}}_{3,5})} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Phi_{\Delta(\widetilde{\mathbb{A}}_{3,5})} = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1)$: –

$P(1,1):\big(R_0^3(1),P(0,2)\big),\ \big(R_\infty^3(1),P(0,4)\big)$
$P(2,1):\big(R_0^3(2),P(0,3)\big),\ \big(R_\infty^3(2),P(0,5)\big),\ \big(R_0^4(1),P(1,2)\big),\ \big(R_\infty^1(1),P(1,4)\big)$
$P(3,1):\big(R_0^2(4),P(0,7)\big),\ \big(R_0^3(3),P(0,8)\big),\ \big(R_0^4(2),P(1,3)\big),\ \big(R_\infty^1(2),P(1,5)\big),\ \big(R_0^5(1),P(2,2)\big)$
$\big(R_\infty^2(1),P(2,4)\big),\ \big(I(1,3),2P(0,6)\big)$
$P(n,1):\big(R_0^{(n-2)\bmod 5+1}(4),P(n-3,7)\big),\ \big(R_0^{(n-1)\bmod 5+1}(3),P(n-3,8)\big),\ \big(R_0^{n\bmod 5+1}(2),P(n-2,3)\big)$
$\big(R_\infty^{(n-3)\bmod 3+1}(2),P(n-2,5)\big),\ \big(R_0^{(n+1)\bmod 5+1}(1),P(n-1,2)\big),\ \big(R_\infty^{(n-2)\bmod 3+1}(1),P(n-1,4)\big)$
$\big(uI,(u+1)P\big),\ n>3$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2)=-1$, for $n\geq 0$.

$P(0,2):\big(R_\infty^3(1),P(0,1)\big)$
$P(1,2):\big(R_0^3(1),P(0,3)\big),\ \big(R_\infty^3(2),P(0,4)\big),\ \big(R_\infty^1(1),P(1,1)\big)$
$P(2,2):\big(R_0^1(4),P(0,6)\big),\ \big(R_0^2(3),P(0,7)\big),\ \big(R_0^3(2),P(0,8)\big),\ \big(R_0^4(1),P(1,3)\big),\ \big(R_\infty^1(2),P(1,4)\big)$
$\big(R_\infty^2(1),P(2,1)\big),\ \big(I(2,8),2P(0,5)\big)$
$P(n,2):\big(R_0^{(n-2)\bmod 5+1}(4),P(n-2,6)\big),\ \big(R_0^{(n-1)\bmod 5+1}(3),P(n-2,7)\big),\ \big(R_0^{n\bmod 5+1}(2),P(n-2,8)\big)$
$\big(R_0^{(n+1)\bmod 5+1}(1),P(n-1,3)\big),\ \big(R_\infty^{(n-2)\bmod 3+1}(2),P(n-1,4)\big),\ \big(R_\infty^{(n-1)\bmod 3+1}(1),P(n,1)\big)$
$\big(uI,(u+1)P\big),\ n>2$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3)=-1$, for $n\geq 0$.

$P(0,3):\big(R_\infty^3(2),P(0,1)\big),\ \big(R_\infty^1(1),P(0,2)\big)$
$P(1,3):\big(R_0^5(4),P(0,5)\big),\ \big(R_0^1(3),P(0,6)\big),\ \big(R_0^2(2),P(0,7)\big),\ \big(R_0^3(1),P(0,8)\big),\ \big(R_\infty^1(2),P(1,1)\big)$
$\big(R_\infty^2(1),P(1,2)\big),\ \big(I(2,7),2P(0,4)\big)$
$P(n,3):\big(R_0^{(n+3)\bmod 5+1}(4),P(n-1,5)\big),\ \big(R_0^{(n-1)\bmod 5+1}(3),P(n-1,6)\big),\ \big(R_0^{n\bmod 5+1}(2),P(n-1,7)\big)$
$\big(R_0^{(n+1)\bmod 5+1}(1),P(n-1,8)\big),\ \big(R_\infty^{(n-1)\bmod 3+1}(2),P(n,1)\big),\ \big(R_\infty^{n\bmod 3+1}(1),P(n,2)\big)$
$\big(uI,(u+1)P\big),\ n>1$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4)=-1$, for $n\geq 0$.

$P(0,4):\big(R_0^3(1),P(0,1)\big)$
$P(1,4):\big(R_0^3(2),P(0,2)\big),\ \big(R_\infty^3(1),P(0,5)\big),\ \big(R_0^4(1),P(1,1)\big)$

$P(2,4) : \big(R_0^3(3), P(0,3)\big),\ \big(R_\infty^3(2), P(0,6)\big),\ \big(R_0^4(2), P(1,2)\big),\ \big(R_\infty^1(1), P(1,5)\big),\ \big(R_0^5(1), P(2,1)\big)$
$P(3,4) : \big(R_0^3(4), P(0,8)\big),\ \big(R_0^4(3), P(1,3)\big),\ \big(R_\infty^1(2), P(1,6)\big),\ \big(R_0^5(2), P(2,2)\big),\ \big(R_\infty^2(1), P(2,5)\big)$
$\big(R_0^1(1), P(3,1)\big),\ \big(I(0,2), 2P(0,7)\big)$
$P(n,4) : \big(R_0^{(n-1) \bmod 5+1}(4), P(n-3,8)\big),\ \big(R_0^{n \bmod 5+1}(3), P(n-2,3)\big),\ \big(R_\infty^{(n-3) \bmod 3+1}(2), P(n-2,6)\big)$
$\big(R_0^{(n+1) \bmod 5+1}(2), P(n-1,2)\big),\ \big(R_\infty^{(n-2) \bmod 3+1}(1), P(n-1,5)\big),\ \big(R_0^{(n-3) \bmod 5+1}(1), P(n,1)\big)$
$\big(uI, (u+1)P\big),\ n > 3$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -1$, for $n \geq 0$.

$P(0,5) : \big(R_0^3(2), P(0,1)\big),\ \big(R_0^4(1), P(0,4)\big)$
$P(1,5) : \big(R_0^3(3), P(0,2)\big),\ \big(R_\infty^3(1), P(0,6)\big),\ \big(R_0^4(2), P(1,1)\big),\ \big(R_0^5(1), P(1,4)\big)$
$P(2,5) : \big(R_0^3(4), P(0,3)\big),\ \big(R_\infty^3(2), P(0,7)\big),\ \big(R_0^4(3), P(1,2)\big),\ \big(R_\infty^1(1), P(1,6)\big),\ \big(R_0^5(2), P(2,1)\big)$
$\big(R_0^1(1), P(2,4)\big)$
$P(3,5) : \big(R_0^4(4), P(1,3)\big),\ \big(R_\infty^1(2), P(1,7)\big),\ \big(R_0^5(3), P(2,2)\big),\ \big(R_\infty^2(1), P(2,6)\big),\ \big(R_0^1(2), P(3,1)\big)$
$\big(R_0^2(1), P(3,4)\big),\ \big(2I(2,6), 3P(0,1)\big)$
$P(n,5) : \big(R_0^{n \bmod 5+1}(4), P(n-2,3)\big),\ \big(R_\infty^{(n-3) \bmod 3+1}(2), P(n-2,7)\big),\ \big(R_0^{(n+1) \bmod 5+1}(3), P(n-1,2)\big)$
$\big(R_\infty^{(n-2) \bmod 3+1}(1), P(n-1,6)\big),\ \big(R_0^{(n-3) \bmod 5+1}(2), P(n,1)\big),\ \big(R_0^{(n-2) \bmod 5+1}(1), P(n,4)\big)$
$\big(uI, (u+1)P\big),\ n > 3$

Modules of the form $P(n,6)$

Defect: $\partial P(n,6) = -1$, for $n \geq 0$.

$P(0,6) : \big(R_0^3(3), P(0,1)\big),\ \big(R_0^4(2), P(0,4)\big),\ \big(R_0^5(1), P(0,5)\big)$
$P(1,6) : \big(R_0^3(4), P(0,2)\big),\ \big(R_\infty^3(1), P(0,7)\big),\ \big(R_0^4(3), P(1,1)\big),\ \big(R_0^5(2), P(1,4)\big),\ \big(R_0^1(1), P(1,5)\big)$
$P(2,6) : \big(R_\infty^3(2), P(0,8)\big),\ \big(R_0^4(4), P(1,2)\big),\ \big(R_\infty^1(1), P(1,7)\big),\ \big(R_0^5(3), P(2,1)\big),\ \big(R_0^1(2), P(2,4)\big)$
$\big(R_0^2(1), P(2,5)\big),\ \big(I(0,4), 2P(0,3)\big)$
$P(n,6) : \big(R_\infty^{n \bmod 3+1}(2), P(n-2,8)\big),\ \big(R_0^{(n+1) \bmod 5+1}(4), P(n-1,2)\big),\ \big(R_\infty^{(n-2) \bmod 3+1}(1), P(n-1,7)\big)$
$\big(R_0^{(n+2) \bmod 5+1}(3), P(n,1)\big),\ \big(R_0^{(n-2) \bmod 5+1}(2), P(n,4)\big),\ \big(R_0^{(n-1) \bmod 5+1}(1), P(n,5)\big)$
$\big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,7)$

Defect: $\partial P(n,7) = -1$, for $n \geq 0$.

$$P(0,7) : \big(R_0^3(4), P(0,1)\big),\ \big(R_0^4(3), P(0,4)\big),\ \big(R_0^5(2), P(0,5)\big),\ \big(R_0^1(1), P(0,6)\big)$$

$$P(1,7) : \big(R_\infty^2(2), P(0,3)\big),\ \big(R_\infty^3(1), P(0,8)\big),\ \big(R_0^4(4), P(1,1)\big),\ \big(R_0^5(3), P(1,4)\big),\ \big(R_0^1(2), P(1,5)\big)$$
$$\big(R_0^2(1), P(1,6)\big),\ \big(I(1,5), 2P(0,2)\big)$$

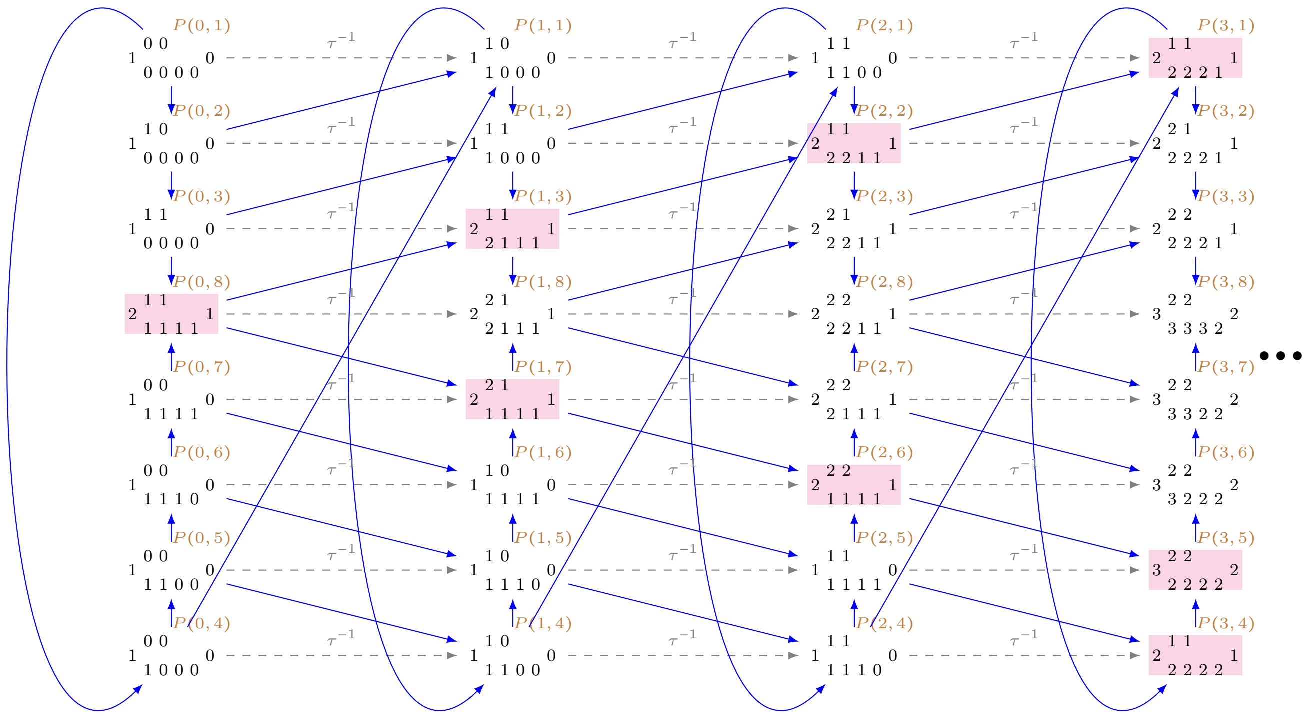
$$P(n,7) : \big(R_\infty^{n \bmod 3+1}(2), P(n-1,3)\big),\ \big(R_\infty^{(n+1) \bmod 3+1}(1), P(n-1,8)\big),\ \big(R_0^{(n+2) \bmod 5+1}(4), P(n,1)\big)$$
$$\big(R_0^{(n+3) \bmod 5+1}(3), P(n,4)\big),\ \big(R_0^{(n-1) \bmod 5+1}(2), P(n,5)\big),\ \big(R_0^{n \bmod 5+1}(1), P(n,6)\big)$$
$$\big(uI, (u+1)P\big),\ n > 1$$

Modules of the form $P(n,8)$

Defect: $\partial P(n,8) = -1$, for $n \geq 0$.

$$P(0,8) : \big(R_\infty^1(2), P(0,2)\big),\ \big(R_\infty^2(1), P(0,3)\big),\ \big(R_0^4(4), P(0,4)\big),\ \big(R_0^5(3), P(0,5)\big),\ \big(R_0^1(2), P(0,6)\big)$$
$$\big(R_0^2(1), P(0,7)\big),\ \big(I(2,6), 2P(0,1)\big)$$

$$P(n,8) : \big(R_\infty^{n \bmod 3+1}(2), P(n,2)\big),\ \big(R_\infty^{(n+1) \bmod 3+1}(1), P(n,3)\big),\ \big(R_0^{(n+3) \bmod 5+1}(4), P(n,4)\big)$$
$$\big(R_0^{(n+4) \bmod 5+1}(3), P(n,5)\big),\ \big(R_0^{n \bmod 5+1}(2), P(n,6)\big),\ \big(R_0^{(n+1) \bmod 5+1}(1), P(n,7)\big)$$
$$\big(uI, (u+1)P\big),\ n > 0$$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$$I(0,1) : \big(I(0,2), R^2_\infty(1)\big),\ \big(I(0,3), R^2_\infty(2)\big),\ \big(I(0,4), R^2_0(1)\big),\ \big(I(0,5), R^2_0(2)\big),\ \big(I(0,6), R^2_0(3)\big)$$
$$\big(I(0,7), R^2_0(4)\big),\ \big(2I(0,8), P(2,5)\big)$$
$$I(n,1) : \big(I(n,2), R^{(-n+1) \bmod 3+1}_\infty(1)\big),\ \big(I(n,3), R^{(-n+1) \bmod 3+1}_\infty(2)\big),\ \big(I(n,4), R^{(-n+1) \bmod 5+1}_0(1)\big)$$
$$\big(I(n,5), R^{(-n+1) \bmod 5+1}_0(2)\big),\ \big(I(n,6), R^{(-n+1) \bmod 5+1}_0(3)\big),\ \big(I(n,7), R^{(-n+1) \bmod 5+1}_0(4)\big)$$
$$\big((v+1)I, vP\big),\ n > 0$$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$$I(0,2) : \big(I(0,3), R^3_\infty(1)\big),\ \big(I(0,8), R^3_\infty(2)\big)$$

$I(1,2):\big(I(0,1),R_0^1(1)\big),\ \big(I(0,4),R_0^1(2)\big),\ \big(I(0,5),R_0^1(3)\big),\ \big(I(0,6),R_0^1(4)\big),\ \big(I(1,3),R_\infty^2(1)\big)$
$\big(I(1,8),R_\infty^2(2)\big),\ \big(2I(0,7),P(2,4)\big)$
$I(n,2):\big(I(n-1,1),R_0^{(-n+1)\text{ mod }5+1}(1)\big),\ \big(I(n-1,4),R_0^{(-n+1)\text{ mod }5+1}(2)\big),\ \big(I(n-1,5),R_0^{(-n+1)\text{ mod }5+1}(3)\big)$
$\big(I(n-1,6),R_0^{(-n+1)\text{ mod }5+1}(4)\big),\ \big(I(n,3),R_\infty^{(-n+2)\text{ mod }3+1}(1)\big),\ \big(I(n,8),R_\infty^{(-n+2)\text{ mod }3+1}(2)\big)$
$\big((v+1)I,vP\big),\ n>1$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3)=1$, for $n\geq 0$.

$I(0,3):\big(I(0,8),R_\infty^1(1)\big)$
$I(1,3):\big(I(0,2),R_0^1(1)\big),\ \big(I(0,7),R_\infty^3(2)\big),\ \big(I(1,8),R_\infty^3(1)\big)$
$I(2,3):\big(I(0,1),R_0^5(2)\big),\ \big(I(0,4),R_0^5(3)\big),\ \big(I(0,5),R_0^5(4)\big),\ \big(I(1,2),R_0^5(1)\big),\ \big(I(1,7),R_\infty^2(2)\big)$
$\big(I(2,8),R_\infty^2(1)\big),\ \big(2I(0,6),P(2,1)\big)$
$I(n,3):\big(I(n-2,1),R_0^{(-n+6)\text{ mod }5+1}(2)\big),\ \big(I(n-2,4),R_0^{(-n+6)\text{ mod }5+1}(3)\big),\ \big(I(n-2,5),R_0^{(-n+6)\text{ mod }5+1}(4)\big)$
$\big(I(n-1,2),R_0^{(-n+6)\text{ mod }5+1}(1)\big),\ \big(I(n-1,7),R_\infty^{(-n+3)\text{ mod }3+1}(2)\big),\ \big(I(n,8),R_\infty^{(-n+3)\text{ mod }3+1}(1)\big)$
$\big((v+1)I,vP\big),\ n>2$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4)=1$, for $n\geq 0$.

$I(0,4):\big(I(0,5),R_0^3(1)\big),\ \big(I(0,6),R_0^3(2)\big),\ \big(I(0,7),R_0^3(3)\big),\ \big(I(0,8),R_0^3(4)\big)$
$I(1,4):\big(I(0,1),R_\infty^1(1)\big),\ \big(I(0,2),R_\infty^1(2)\big),\ \big(I(1,5),R_0^2(1)\big),\ \big(I(1,6),R_0^2(2)\big),\ \big(I(1,7),R_0^2(3)\big)$
$\big(I(1,8),R_0^2(4)\big),\ \big(2I(0,3),P(1,6)\big)$
$I(n,4):\big(I(n-1,1),R_\infty^{(-n+1)\text{ mod }3+1}(1)\big),\ \big(I(n-1,2),R_\infty^{(-n+1)\text{ mod }3+1}(2)\big),\ \big(I(n,5),R_0^{(-n+2)\text{ mod }5+1}(1)\big)$
$\big(I(n,6),R_0^{(-n+2)\text{ mod }5+1}(2)\big),\ \big(I(n,7),R_0^{(-n+2)\text{ mod }5+1}(3)\big),\ \big(I(n,8),R_0^{(-n+2)\text{ mod }5+1}(4)\big)$
$\big((v+1)I,vP\big),\ n>1$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5)=1$, for $n\geq 0$.

$I(0,5):\big(I(0,6),R_0^4(1)\big),\ \big(I(0,7),R_0^4(2)\big),\ \big(I(0,8),R_0^4(3)\big)$
$I(1,5):\big(I(0,3),R_0^3(4)\big),\ \big(I(0,4),R_\infty^1(1)\big),\ \big(I(1,6),R_0^3(1)\big),\ \big(I(1,7),R_0^3(2)\big),\ \big(I(1,8),R_0^3(3)\big)$
$I(2,5):\big(I(0,1),R_\infty^3(2)\big),\ \big(I(1,3),R_0^2(4)\big),\ \big(I(1,4),R_\infty^3(1)\big),\ \big(I(2,6),R_0^2(1)\big),\ \big(I(2,7),R_0^2(2)\big)$
$\big(I(2,8),R_0^2(3)\big),\ \big(2I(0,2),P(0,7)\big)$

$I(n,5): \big(I(n-2,1), R_\infty^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,3), R_0^{(-n+3) \bmod 5+1}(4)\big),\ \big(I(n-1,4), R_\infty^{(-n+4) \bmod 3+1}(1)\big)$
$\big(I(n,6), R_0^{(-n+3) \bmod 5+1}(1)\big),\ \big(I(n,7), R_0^{(-n+3) \bmod 5+1}(2)\big),\ \big(I(n,8), R_0^{(-n+3) \bmod 5+1}(3)\big)$
$\big((v+1)I, vP\big),\ n>2$

Modules of the form $I(n,6)$

Defect: $\partial I(n,6)=1$, for $n \geq 0$.

$I(0,6): \big(I(0,7), R_0^5(1)\big),\ \big(I(0,8), R_0^5(2)\big)$
$I(1,6): \big(I(0,3), R_0^4(3)\big),\ \big(I(0,5), R_\infty^1(1)\big),\ \big(I(1,7), R_0^4(1)\big),\ \big(I(1,8), R_0^4(2)\big)$
$I(2,6): \big(I(0,2), R_0^3(4)\big),\ \big(I(0,4), R_\infty^3(2)\big),\ \big(I(1,3), R_0^3(3)\big),\ \big(I(1,5), R_\infty^3(1)\big),\ \big(I(2,7), R_0^3(1)\big)$
$\big(I(2,8), R_0^3(2)\big)$
$I(3,6): \big(I(1,2), R_0^2(4)\big),\ \big(I(1,4), R_\infty^2(2)\big),\ \big(I(2,3), R_0^2(3)\big),\ \big(I(2,5), R_\infty^2(1)\big),\ \big(I(3,7), R_0^2(1)\big)$
$\big(I(3,8), R_0^2(2)\big),\ \big(3I(0,8), 2P(2,5)\big)$
$I(n,6): \big(I(n-2,2), R_0^{(-n+4) \bmod 5+1}(4)\big),\ \big(I(n-2,4), R_\infty^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,3), R_0^{(-n+4) \bmod 5+1}(3)\big)$
$\big(I(n-1,5), R_\infty^{(-n+4) \bmod 3+1}(1)\big),\ \big(I(n,7), R_0^{(-n+4) \bmod 5+1}(1)\big),\ \big(I(n,8), R_0^{(-n+4) \bmod 5+1}(2)\big)$
$\big((v+1)I, vP\big),\ n>3$

Modules of the form $I(n,7)$

Defect: $\partial I(n,7)=1$, for $n \geq 0$.

$I(0,7): \big(I(0,8), R_0^1(1)\big)$
$I(1,7): \big(I(0,3), R_0^5(2)\big),\ \big(I(0,6), R_\infty^1(1)\big),\ \big(I(1,8), R_0^5(1)\big)$
$I(2,7): \big(I(0,2), R_0^4(3)\big),\ \big(I(0,5), R_\infty^3(2)\big),\ \big(I(1,3), R_0^4(2)\big),\ \big(I(1,6), R_\infty^3(1)\big),\ \big(I(2,8), R_0^4(1)\big)$
$I(3,7): \big(I(0,1), R_0^3(4)\big),\ \big(I(1,2), R_0^3(3)\big),\ \big(I(1,5), R_\infty^2(2)\big),\ \big(I(2,3), R_0^3(2)\big),\ \big(I(2,6), R_\infty^2(1)\big)$
$\big(I(3,8), R_0^3(1)\big),\ \big(2I(0,4), P(0,3)\big)$
$I(n,7): \big(I(n-3,1), R_0^{(-n+5) \bmod 5+1}(4)\big),\ \big(I(n-2,2), R_0^{(-n+5) \bmod 5+1}(3)\big),\ \big(I(n-2,5), R_\infty^{(-n+4) \bmod 3+1}(2)\big)$
$\big(I(n-1,3), R_0^{(-n+5) \bmod 5+1}(2)\big),\ \big(I(n-1,6), R_\infty^{(-n+4) \bmod 3+1}(1)\big),\ \big(I(n,8), R_0^{(-n+5) \bmod 5+1}(1)\big)$
$\big((v+1)I, vP\big),\ n>3$

Modules of the form $I(n,8)$

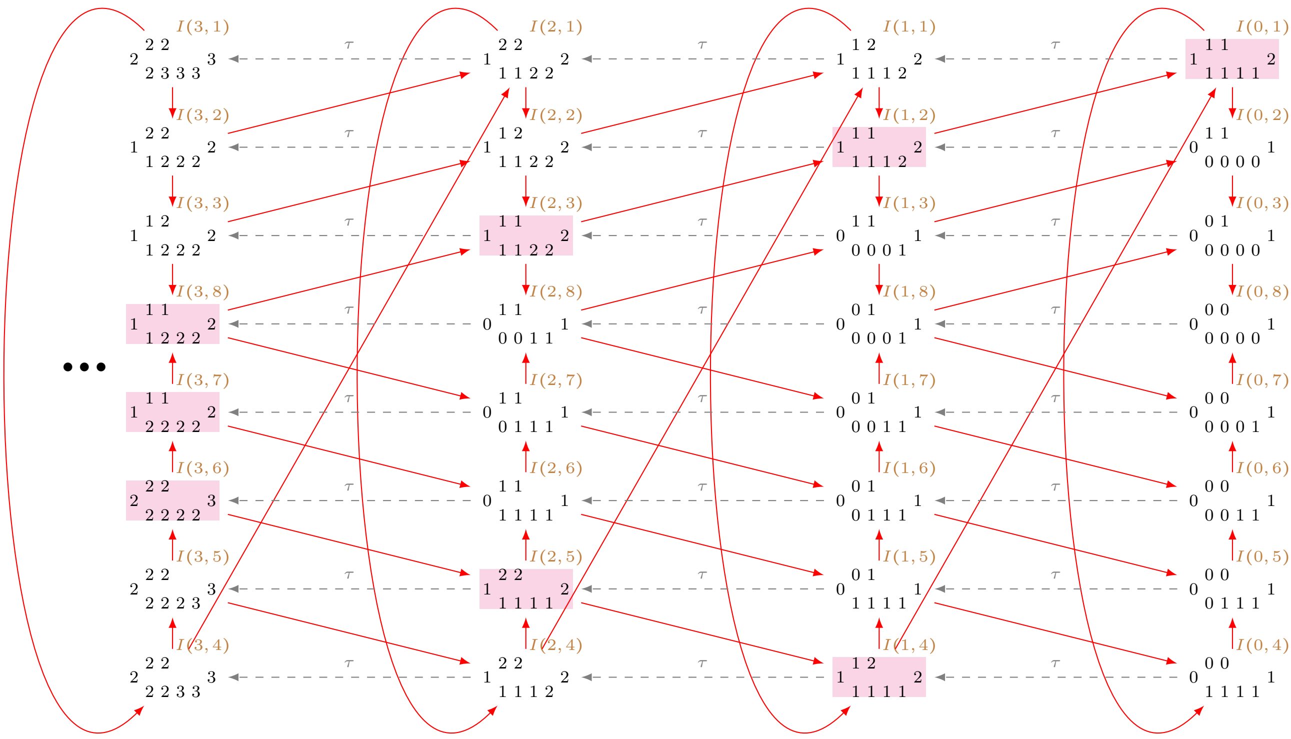
Defect: $\partial I(n,8)=1$, for $n \geq 0$.

$I(0,8): -$
$I(1,8): \big(I(0,3), R_0^1(1)\big),\ \big(I(0,7), R_\infty^1(1)\big)$

$$I(2,8): \big(I(0,2), R_0^5(2)\big),\ \big(I(0,6), R_\infty^3(2)\big),\ \big(I(1,3), R_0^5(1)\big),\ \big(I(1,7), R_\infty^3(1)\big)$$

$$I(3,8): \big(I(0,1), R_0^4(3)\big),\ \big(I(0,4), R_0^4(4)\big),\ \big(I(1,2), R_0^4(2)\big),\ \big(I(1,6), R_\infty^2(2)\big),\ \big(I(2,3), R_0^4(1)\big)$$
$$\big(I(2,7), R_\infty^2(1)\big),\ \big(2I(0,5), P(1,2)\big)$$

$$I(n,8): \big(I(n-3,1), R_0^{(-n+6) \bmod 5+1}(3)\big),\ \big(I(n-3,4), R_0^{(-n+6) \bmod 5+1}(4)\big),\ \big(I(n-2,2), R_0^{(-n+6) \bmod 5+1}(2)\big)$$
$$\big(I(n-2,6), R_\infty^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,3), R_0^{(-n+6) \bmod 5+1}(1)\big),\ \big(I(n-1,7), R_\infty^{(-n+4) \bmod 3+1}(1)\big)$$
$$\big((v+1)I, vP\big),\ n>3$$



Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{3,5})}$

$R_0^1(1)$: –

$R_0^1(2) : \big(R_0^2(1), R_0^1(1)\big),\ \big(I(1,3), P(0,1)\big),\ \big(I(1,8), P(0,2)\big),\ \big(I(0,7), P(0,3)\big)$

$R_0^2(1) : \big(I(0,2), P(0,1)\big),\ \big(I(0,3), P(0,2)\big),\ \big(I(0,8), P(0,3)\big)$

$R_0^2(2) : \big(R_0^3(1), R_0^2(1)\big),\ \big(I(0,2), P(0,4)\big),\ \big(I(0,3), P(1,1)\big),\ \big(I(0,8), P(1,2)\big)$

$R_0^3(1)$: –

$R_0^3(2) : \big(R_0^4(1), R_0^3(1)\big)$

$R_0^4(1)$: –

$R_0^4(2) : \big(R_0^5(1), R_0^4(1)\big)$

$R_0^5(1)$: –

$R_0^5(2) : \big(R_0^1(1), R_0^5(1)\big)$

$R_0^5(3) : \big(R_0^1(2), R_0^5(1)\big),\ \big(R_0^2(1), R_0^5(2)\big),\ \big(I(2,8), P(0,1)\big),\ \big(I(1,7), P(0,2)\big),\ \big(I(0,6), P(0,3)\big)$

$R_0^5(4) : \big(R_0^1(3), R_0^5(1)\big),\ \big(R_0^2(2), R_0^5(2)\big),\ \big(R_0^3(1), R_0^5(3)\big),\ \big(I(2,8), P(0,4)\big),\ \big(I(1,7), P(1,1)\big)$
$\big(I(0,6), P(1,2)\big)$

$R_0^1(3) : \big(R_0^2(2), R_0^1(1)\big),\ \big(R_0^3(1), R_0^1(2)\big),\ \big(I(1,3), P(0,4)\big),\ \big(I(1,8), P(1,1)\big),\ \big(I(0,7), P(1,2)\big)$

$R_0^1(4) : \big(R_0^2(3), R_0^1(1)\big),\ \big(R_0^3(2), R_0^1(2)\big),\ \big(R_0^4(1), R_0^1(3)\big),\ \big(I(1,3), P(0,5)\big),\ \big(I(1,8), P(1,4)\big)$
$\big(I(0,7), P(2,1)\big)$

$R_0^2(3) : \big(R_0^3(2), R_0^2(1)\big),\ \big(R_0^4(1), R_0^2(2)\big),\ \big(I(0,2), P(0,5)\big),\ \big(I(0,3), P(1,4)\big),\ \big(I(0,8), P(2,1)\big)$

$R_0^2(4) : \big(R_0^3(3), R_0^2(1)\big),\ \big(R_0^4(2), R_0^2(2)\big),\ \big(R_0^5(1), R_0^2(3)\big),\ \big(I(0,2), P(0,6)\big),\ \big(I(0,3), P(1,5)\big)$
$\big(I(0,8), P(2,4)\big)$

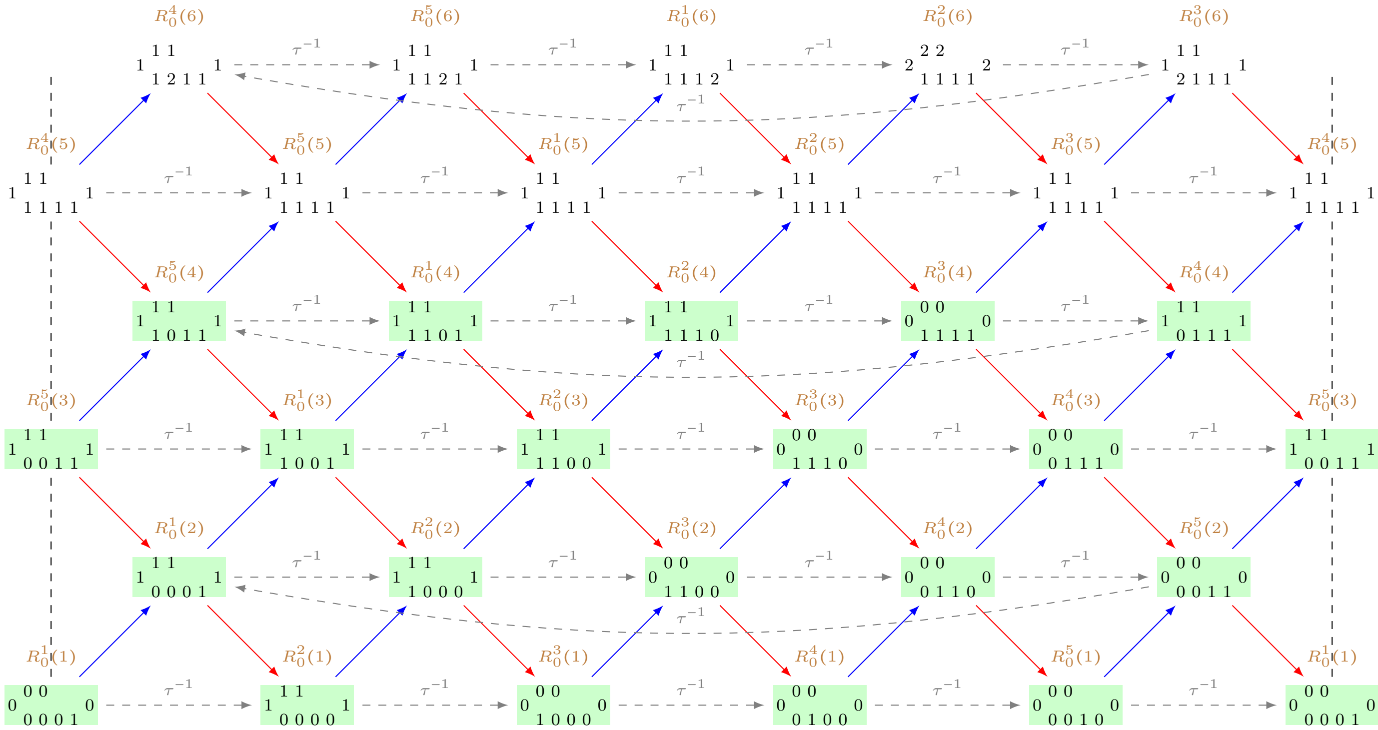
$R_0^3(3) : \big(R_0^4(2), R_0^3(1)\big),\ \big(R_0^5(1), R_0^3(2)\big)$

$R_0^3(4) : \big(R_0^4(3), R_0^3(1)\big),\ \big(R_0^5(2), R_0^3(2)\big),\ \big(R_0^1(1), R_0^3(3)\big)$

$R_0^4(3) : \big(R_0^5(2), R_0^4(1)\big),\ \big(R_0^1(1), R_0^4(2)\big)$

$R_0^4(4) : \big(R_0^5(3), R_0^4(1)\big),\ \big(R_0^1(2), R_0^4(2)\big),\ \big(R_0^2(1), R_0^4(3)\big),\ \big(I(2,7), P(0,1)\big),\ \big(I(1,6), P(0,2)\big)$
$\big(I(0,5), P(0,3)\big)$

$\mathcal{T}_0^{\Delta(\widetilde{\mathbb{A}}_{3,5})}$



The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{3,5})}$

$R^1_\infty(1)$: –

$R^1_\infty(2) : \big(R^2_\infty(1), R^1_\infty(1)\big),\ \big(I(1,5), P(0,1)\big),\ \big(I(1,6), P(0,4)\big),\ \big(I(1,7), P(0,5)\big),\ \big(I(1,8), P(0,6)\big)$
$\big(I(0,3), P(0,7)\big)$

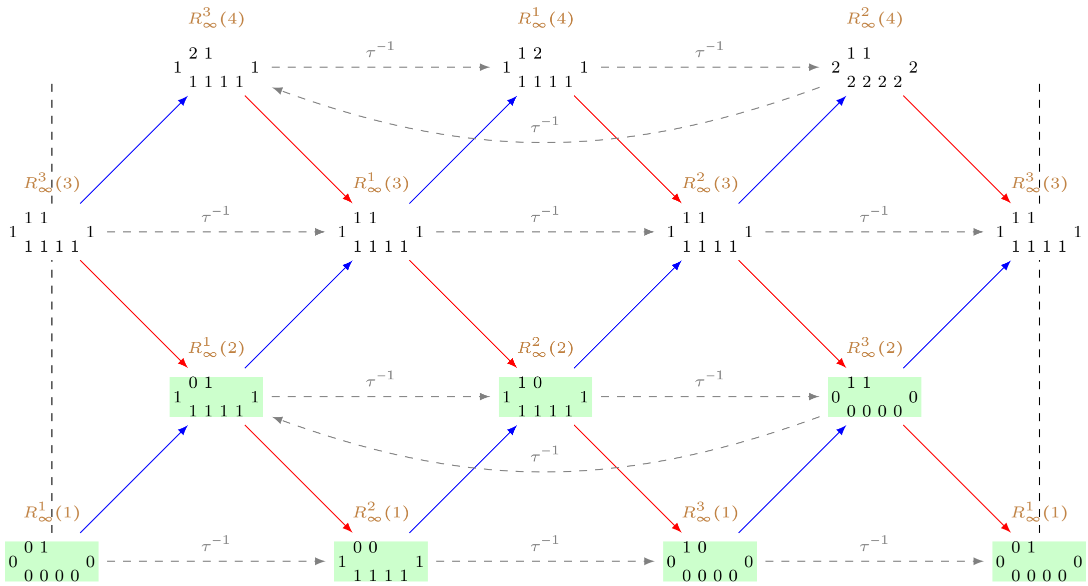
$R^2_\infty(1) : \big(I(0,4), P(0,1)\big),\ \big(I(0,5), P(0,4)\big),\ \big(I(0,6), P(0,5)\big),\ \big(I(0,7), P(0,6)\big),\ \big(I(0,8), P(0,7)\big)$

$R^2_\infty(2) : \big(R^3_\infty(1), R^2_\infty(1)\big),\ \big(I(0,4), P(0,2)\big),\ \big(I(0,5), P(1,1)\big),\ \big(I(0,6), P(1,4)\big),\ \big(I(0,7), P(1,5)\big)$
$\big(I(0,8), P(1,6)\big)$

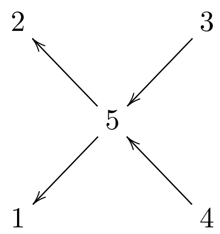
$R^3_\infty(1)$: –

$R^3_\infty(2) : \big(R^1_\infty(1), R^3_\infty(1)\big)$

$\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{A}}_{3,5})}$



A.10 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{D}}_4)$ – $\delta = {}^{1}_{1}\,2\,{}^{1}_{1}$



$$C_{\Delta(\widetilde{\mathbb{D}}_4)} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{D}}_4)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1)$: –
$P(1,1) : \big(R_1^1(1), P(0,2)\big)$
$P(2,1) : \big(R_0^2(1), P(0,3)\big),\ \big(R_\infty^2(1), P(0,4)\big),\ \big(R_1^2(1), P(1,2)\big),\ \big(I(1,1), 2P(0,1)\big)$
$P(n,1) : \big(R_0^{(n-1) \bmod 2+1}(1), P(n-2,3)\big),\ \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n-2,4)\big),\ \big(R_1^{(n-1) \bmod 2+1}(1), P(n-1,2)\big)$
$\big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$P(0,2)$: –
$P(1,2) : \big(R_1^1(1), P(0,1)\big)$
$P(2,2) : \big(R_\infty^1(1), P(0,3)\big),\ \big(R_0^1(1), P(0,4)\big),\ \big(R_1^2(1), P(1,1)\big),\ \big(I(1,2), 2P(0,2)\big)$
$P(n,2) : \big(R_\infty^{(n-2) \bmod 2+1}(1), P(n-2,3)\big),\ \big(R_0^{(n-2) \bmod 2+1}(1), P(n-2,4)\big),\ \big(R_1^{(n-1) \bmod 2+1}(1), P(n-1,1)\big)$
$\big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$P(0,3) : \big(R_0^1(1), P(0,1)\big),\ \big(R_\infty^2(1), P(0,2)\big),\ \big(I(0,3), P(0,5)\big)$
$P(1,3) : \big(R_1^1(1), P(0,4)\big),\ \big(R_0^2(1), P(1,1)\big),\ \big(R_\infty^1(1), P(1,2)\big)$
$P(2,3) : \big(R_1^2(1), P(1,4)\big),\ \big(R_0^1(1), P(2,1)\big),\ \big(R_\infty^2(1), P(2,2)\big),\ \big(I(1,3), 2P(0,3)\big)$
$P(n,3) : \big(R_1^{(n-1) \bmod 2+1}(1), P(n-1,4)\big),\ \big(R_0^{(n-2) \bmod 2+1}(1), P(n,1)\big),\ \big(R_\infty^{(n-1) \bmod 2+1}(1), P(n,2)\big)$
$\big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,4)$

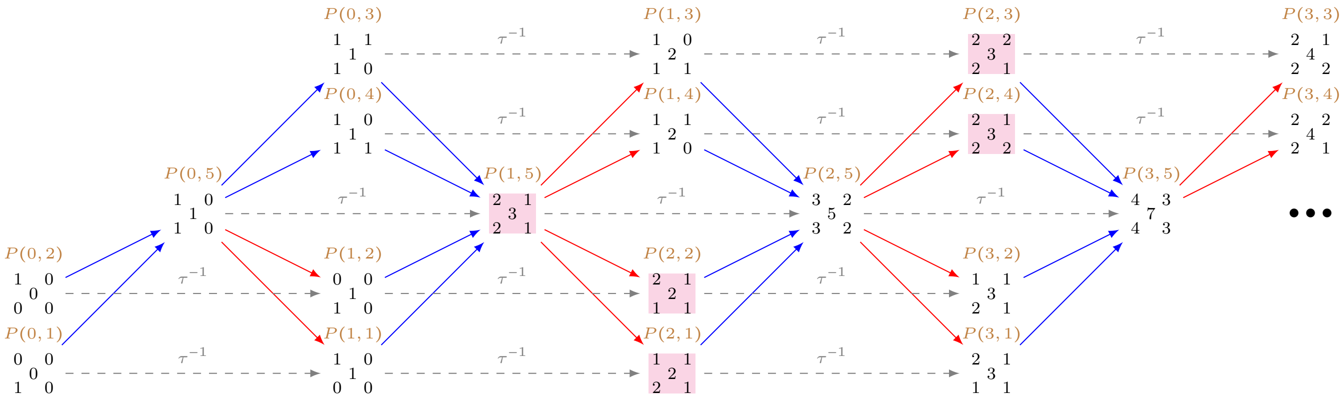
Defect: $\partial P(n,4) = -1$, for $n \geq 0$.

$P(0,4) : \big(R_\infty^1(1), P(0,1)\big),\ \big(R_0^2(1), P(0,2)\big),\ \big(I(0,4), P(0,5)\big)$
$P(1,4) : \big(R_1^1(1), P(0,3)\big),\ \big(R_\infty^2(1), P(1,1)\big),\ \big(R_0^1(1), P(1,2)\big)$
$P(2,4) : \big(R_1^2(1), P(1,3)\big),\ \big(R_\infty^1(1), P(2,1)\big),\ \big(R_0^2(1), P(2,2)\big),\ \big(I(1,4), 2P(0,4)\big)$
$P(n,4) : \big(R_1^{(n-1) \bmod 2+1}(1), P(n-1,3)\big),\ \big(R_\infty^{(n-2) \bmod 2+1}(1), P(n,1)\big),\ \big(R_0^{(n-1) \bmod 2+1}(1), P(n,2)\big)$
$\big(uI, (u+1)P\big),\ n > 2$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -2$, for $n \geq 0$.

$P(0,5) : \big(P(1,1), P(0,1)\big),\ \big(P(1,2), P(0,2)\big)$
$P(1,5) : \big(P(1,3), P(0,3)\big),\ \big(P(1,4), P(0,4)\big),\ \big(P(2,1), P(1,1)\big),\ \big(P(2,2), P(1,2)\big)$
$P(n,5) : \big(P(n,3), P(n-1,3)\big),\ \big(P(n,4), P(n-1,4)\big),\ \big(P(n+1,1), P(n,1)\big)$
$\big(P(n+1,2), P(n,2)\big),\ n > 1$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$I(0,1) : \big(I(0,3), R_0^2(1)\big),\ \big(I(0,4), R_\infty^2(1)\big),\ \big(I(0,5), P(0,1)\big)$
$I(1,1) : \big(I(0,2), R_1^1(1)\big),\ \big(I(1,3), R_0^1(1)\big),\ \big(I(1,4), R_\infty^1(1)\big)$
$I(2,1) : \big(I(1,2), R_1^2(1)\big),\ \big(I(2,3), R_0^2(1)\big),\ \big(I(2,4), R_\infty^2(1)\big),\ \big(2I(0,1), P(1,1)\big)$
$I(n,1) : \big(I(n-1,2), R_1^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n,3), R_0^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n,4), R_\infty^{(-n+3) \bmod 2+1}(1)\big)$
$\big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$I(0,2) : \big(I(0,3), R_\infty^1(1)\big),\ \big(I(0,4), R_0^1(1)\big),\ \big(I(0,5), P(0,2)\big)$
$I(1,2) : \big(I(0,1), R_1^1(1)\big),\ \big(I(1,3), R_\infty^2(1)\big),\ \big(I(1,4), R_0^2(1)\big)$
$I(2,2) : \big(I(1,1), R_1^2(1)\big),\ \big(I(2,3), R_\infty^1(1)\big),\ \big(I(2,4), R_0^1(1)\big),\ \big(2I(0,2), P(1,2)\big)$
$I(n,2) : \big(I(n-1,1), R_1^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n,3), R_\infty^{(-n+2) \bmod 2+1}(1)\big),\ \big(I(n,4), R_0^{(-n+2) \bmod 2+1}(1)\big)$
$\big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$I(0,3) : -$
$I(1,3) : \big(I(0,4), R_1^1(1)\big)$
$I(2,3) : \big(I(0,1), R_0^1(1)\big),\ \big(I(0,2), R_\infty^2(1)\big),\ \big(I(1,4), R_1^2(1)\big),\ \big(2I(0,3), P(1,3)\big)$

$I(n,3) : \big(I(n-2,1), R_0^{(-n+2) \bmod 2+1}(1)\big),\ \big(I(n-2,2), R_\infty^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n-1,4), R_1^{(-n+3) \bmod 2+1}(1)\big)$
$\quad \big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,4)$

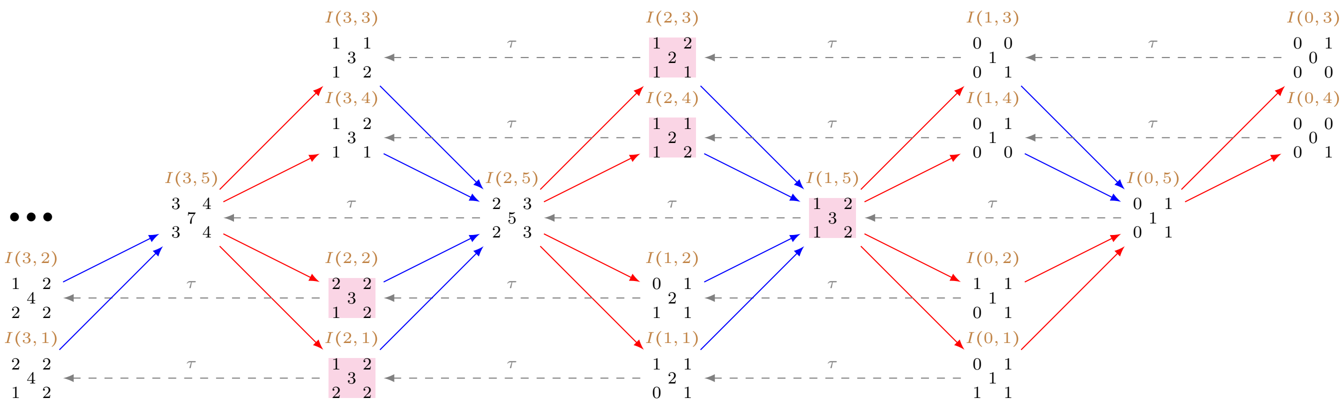
Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$I(0,4) : -$
$I(1,4) : \big(I(0,3), R_1^1(1)\big)$
$I(2,4) : \big(I(0,1), R_\infty^1(1)\big),\ \big(I(0,2), R_0^2(1)\big),\ \big(I(1,3), R_1^2(1)\big),\ \big(2I(0,4), P(1,4)\big)$
$I(n,4) : \big(I(n-2,1), R_\infty^{(-n+2) \bmod 2+1}(1)\big),\ \big(I(n-2,2), R_0^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n-1,3), R_1^{(-n+3) \bmod 2+1}(1)\big)$
$\quad \big((v+1)I, vP\big),\ n > 2$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 2$, for $n \geq 0$.

$I(0,5) : \big(I(0,3), I(1,3)\big),\ \big(I(0,4), I(1,4)\big)$
$I(1,5) : \big(I(0,1), I(1,1)\big),\ \big(I(0,2), I(1,2)\big),\ \big(I(1,3), I(2,3)\big),\ \big(I(1,4), I(2,4)\big)$
$I(n,5) : \big(I(n-1,1), I(n,1)\big),\ \big(I(n-1,2), I(n,2)\big),\ \big(I(n,3), I(n+1,3)\big)$
$\quad \big(I(n,4), I(n+1,4)\big),\ n > 1$

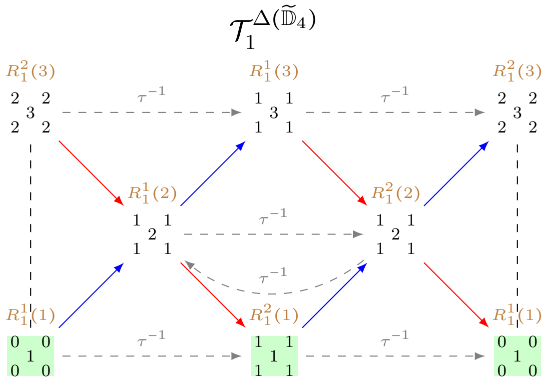


Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_4)}$

$R_1^1(1)$: –

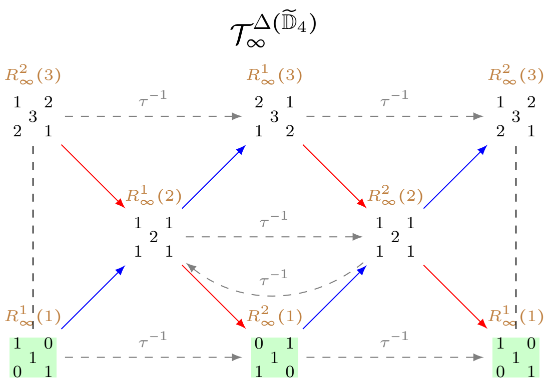
$R_1^2(1) : \big(I(0,2), P(0,1)\big),\ \big(I(0,1), P(0,2)\big),\ \big(I(0,4), P(0,3)\big),\ \big(I(0,3), P(0,4)\big)$



The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_4)}$

$R_\infty^1(1) : \big(I(1,3), P(0,2)\big),\ \big(I(0,4), P(1,1)\big)$

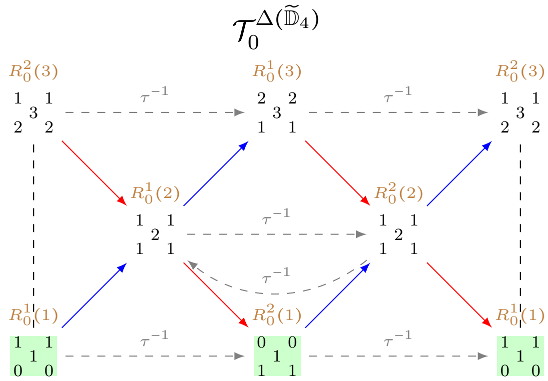
$R_\infty^2(1) : \big(I(1,4), P(0,1)\big),\ \big(I(0,3), P(1,2)\big)$



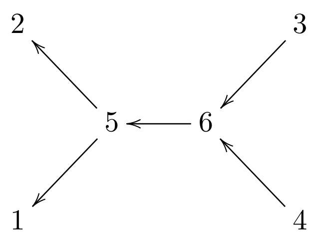
The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_4)}$

$R_0^1(1) : \big(I(1,4), P(0,2)\big),\ \big(I(0,3), P(1,1)\big)$

$R_0^2(1) : \big(I(1,3), P(0,1)\big),\ \big(I(0,4), P(1,2)\big)$



A.11 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{D}}_5)$ – $\delta = {}^{1}_{1}\,2\,2\,{}^{1}_{1}$



$$C_{\Delta(\widetilde{\mathbb{D}}_5)} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{D}}_5)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1) : -$
$P(1,1) : \big(R_1^3(1), P(0,2)\big)$
$P(2,1) : \big(R_1^3(2), P(0,1)\big),\ \big(R_1^1(1), P(1,2)\big)$
$P(3,1) : \big(R_\infty^1(1), P(0,3)\big),\ \big(R_0^1(1), P(0,4)\big),\ \big(R_1^1(2), P(1,1)\big),\ \big(R_1^2(1), P(2,2)\big),\ \big(I(2,1), 2P(0,2)\big)$
$P(n,1) : \big(R_\infty^{(n-3) \bmod 2+1}(1), P(n-3,3)\big),\ \big(R_0^{(n-3) \bmod 2+1}(1), P(n-3,4)\big),\ \big(R_1^{(n-3) \bmod 3+1}(2), P(n-2,1)\big)$
$\big(R_1^{(n-2) \bmod 3+1}(1), P(n-1,2)\big),\ \big(uI, (u+1)P\big),\ n > 3$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$P(0,2):-$
$P(1,2):\big(R_1^3(1),P(0,1)\big)$
$P(2,2):\big(R_1^3(2),P(0,2)\big),\ \big(R_1^1(1),P(1,1)\big)$
$P(3,2):\big(R_0^2(1),P(0,3)\big),\ \big(R_\infty^2(1),P(0,4)\big),\ \big(R_1^1(2),P(1,2)\big),\ \big(R_1^2(1),P(2,1)\big),\ \big(I(2,2),2P(0,1)\big)$
$P(n,2):\big(R_0^{(n-2)\bmod 2+1}(1),P(n-3,3)\big),\ \big(R_\infty^{(n-2)\bmod 2+1}(1),P(n-3,4)\big),\ \big(R_1^{(n-3)\bmod 3+1}(2),P(n-2,2)\big)$
$\big(R_1^{(n-2)\bmod 3+1}(1),P(n-1,1)\big),\ \big(uI,(u+1)P\big),\ n>3$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3)=-1$, for $n\geq 0$.

$P(0,3):\big(R_0^1(1),P(0,1)\big),\ \big(R_\infty^2(1),P(0,2)\big),\ \big(I(1,4),P(0,5)\big),\ \big(I(0,3),P(0,6)\big)$
$P(1,3):\big(R_1^3(1),P(0,4)\big),\ \big(R_0^2(1),P(1,1)\big),\ \big(R_\infty^1(1),P(1,2)\big),\ \big(I(0,4),P(1,5)\big)$
$P(2,3):\big(R_1^3(2),P(0,3)\big),\ \big(R_1^1(1),P(1,4)\big),\ \big(R_0^1(1),P(2,1)\big),\ \big(R_\infty^2(1),P(2,2)\big)$
$P(3,3):\big(R_1^1(2),P(1,3)\big),\ \big(R_1^2(1),P(2,4)\big),\ \big(R_0^2(1),P(3,1)\big),\ \big(R_\infty^1(1),P(3,2)\big),\ \big(I(2,3),2P(0,4)\big)$
$P(n,3):\big(R_1^{(n-3)\bmod 3+1}(2),P(n-2,3)\big),\ \big(R_1^{(n-2)\bmod 3+1}(1),P(n-1,4)\big),\ \big(R_0^{(n-2)\bmod 2+1}(1),P(n,1)\big)$
$\big(R_\infty^{(n-3)\bmod 2+1}(1),P(n,2)\big),\ \big(uI,(u+1)P\big),\ n>3$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4)=-1$, for $n\geq 0$.

$P(0,4):\big(R_\infty^1(1),P(0,1)\big),\ \big(R_0^2(1),P(0,2)\big),\ \big(I(1,3),P(0,5)\big),\ \big(I(0,4),P(0,6)\big)$
$P(1,4):\big(R_1^3(1),P(0,3)\big),\ \big(R_\infty^2(1),P(1,1)\big),\ \big(R_0^1(1),P(1,2)\big),\ \big(I(0,3),P(1,5)\big)$
$P(2,4):\big(R_1^3(2),P(0,4)\big),\ \big(R_1^1(1),P(1,3)\big),\ \big(R_\infty^1(1),P(2,1)\big),\ \big(R_0^2(1),P(2,2)\big)$
$P(3,4):\big(R_1^1(2),P(1,4)\big),\ \big(R_1^2(1),P(2,3)\big),\ \big(R_\infty^2(1),P(3,1)\big),\ \big(R_0^1(1),P(3,2)\big),\ \big(I(2,4),2P(0,3)\big)$
$P(n,4):\big(R_1^{(n-3)\bmod 3+1}(2),P(n-2,4)\big),\ \big(R_1^{(n-2)\bmod 3+1}(1),P(n-1,3)\big),\ \big(R_\infty^{(n-2)\bmod 2+1}(1),P(n,1)\big)$
$\big(R_0^{(n-3)\bmod 2+1}(1),P(n,2)\big),\ \big(uI,(u+1)P\big),\ n>3$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5)=-2$, for $n\geq 0$.

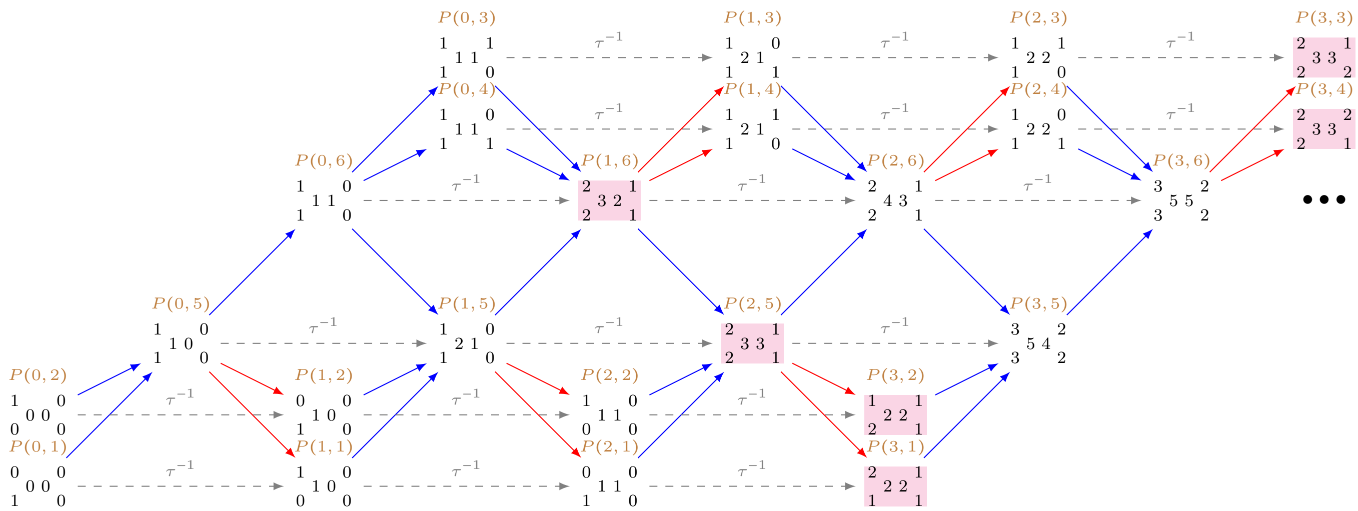
$P(0,5):\big(P(1,1),P(0,1)\big),\ \big(P(1,2),P(0,2)\big)$
$P(1,5):\big(R_1^3(1),P(0,6)\big),\ \big(P(2,1),P(1,1)\big),\ \big(P(2,2),P(1,2)\big)$
$P(2,5):\big(P(2,4),P(0,3)\big),\ \big(P(2,3),P(0,4)\big),\ \big(R_1^1(1),P(1,6)\big),\ \big(P(3,1),P(2,1)\big),\ \big(P(3,2),P(2,2)\big)$
$P(n,5):\big(P(n,4),P(n-2,3)\big),\ \big(P(n,3),P(n-2,4)\big),\ \big(R_1^{(n-2)\bmod 3+1}(1),P(n-1,6)\big)$

$$\big(P(n+1,1),P(n,1)\big),\ \big(P(n+1,2),P(n,2)\big),\ n>2$$

Modules of the form $P(n,6)$

Defect: $\partial P(n,6)=-2$, for $n\geq 0$.

$P(0,6):\big(P(2,2),P(0,1)\big),\ \big(P(2,1),P(0,2)\big),\ \big(R_1^1(1),P(0,5)\big)$
$P(1,6):\big(P(1,3),P(0,3)\big),\ \big(P(1,4),P(0,4)\big),\ \big(P(3,2),P(1,1)\big),\ \big(P(3,1),P(1,2)\big),\ \big(R_1^2(1),P(1,5)\big)$
$P(n,6):\big(P(n,3),P(n-1,3)\big),\ \big(P(n,4),P(n-1,4)\big),\ \big(P(n+2,2),P(n,1)\big)$
$\big(P(n+2,1),P(n,2)\big),\ \big(R_1^{n \bmod 3+1}(1),P(n,5)\big),\ n>1$



Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1)=1$, for $n\geq 0$.

$I(0,1):\big(I(0,3),R_0^2(1)\big),\ \big(I(0,4),R_\infty^2(1)\big),\ \big(I(0,5),P(0,1)\big),\ \big(I(0,6),P(1,2)\big)$
$I(1,1):\big(I(0,2),R_1^1(1)\big),\ \big(I(1,3),R_0^1(1)\big),\ \big(I(1,4),R_\infty^1(1)\big),\ \big(I(1,6),P(0,2)\big)$
$I(2,1):\big(I(0,1),R_1^3(2)\big),\ \big(I(1,2),R_1^3(1)\big),\ \big(I(2,3),R_0^2(1)\big),\ \big(I(2,4),R_\infty^2(1)\big)$
$I(3,1):\big(I(1,1),R_1^2(2)\big),\ \big(I(2,2),R_1^2(1)\big),\ \big(I(3,3),R_0^1(1)\big),\ \big(I(3,4),R_\infty^1(1)\big),\ \big(2I(0,2),P(2,1)\big)$
$I(n,1):\big(I(n-2,1),R_1^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,2),R_1^{(-n+4) \bmod 3+1}(1)\big),\ \big(I(n,3),R_0^{(-n+3) \bmod 2+1}(1)\big)$
$\big(I(n,4),R_\infty^{(-n+3) \bmod 2+1}(1)\big),\ \big((v+1)I,vP\big),\ n>3$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$$I(0,2) : \big(I(0,3), R^1_\infty(1)\big),\ \big(I(0,4), R^1_0(1)\big),\ \big(I(0,5), P(0,2)\big),\ \big(I(0,6), P(1,1)\big)$$
$$I(1,2) : \big(I(0,1), R^1_1(1)\big),\ \big(I(1,3), R^2_\infty(1)\big),\ \big(I(1,4), R^2_0(1)\big),\ \big(I(1,6), P(0,1)\big)$$
$$I(2,2) : \big(I(0,2), R^3_1(2)\big),\ \big(I(1,1), R^3_1(1)\big),\ \big(I(2,3), R^1_\infty(1)\big),\ \big(I(2,4), R^1_0(1)\big)$$
$$I(3,2) : \big(I(1,2), R^2_1(2)\big),\ \big(I(2,1), R^2_1(1)\big),\ \big(I(3,3), R^2_\infty(1)\big),\ \big(I(3,4), R^2_0(1)\big),\ \big(2I(0,1), P(2,2)\big)$$
$$I(n,2) : \big(I(n-2,2), R_1^{(-n+4) \bmod 3+1}(2)\big),\ \big(I(n-1,1), R_1^{(-n+4) \bmod 3+1}(1)\big),\ \big(I(n,3), R_\infty^{(-n+4) \bmod 2+1}(1)\big)$$
$$\big(I(n,4), R_0^{(-n+4) \bmod 2+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 3$$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$$I(0,3) : -$$
$$I(1,3) : \big(I(0,4), R^1_1(1)\big)$$
$$I(2,3) : \big(I(0,3), R^3_1(2)\big),\ \big(I(1,4), R^3_1(1)\big)$$
$$I(3,3) : \big(I(0,1), R^1_\infty(1)\big),\ \big(I(0,2), R^2_0(1)\big),\ \big(I(1,3), R^2_1(2)\big),\ \big(I(2,4), R^2_1(1)\big),\ \big(2I(0,4), P(2,3)\big)$$
$$I(n,3) : \big(I(n-3,1), R_\infty^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n-3,2), R_0^{(-n+4) \bmod 2+1}(1)\big),\ \big(I(n-2,3), R_1^{(-n+4) \bmod 3+1}(2)\big)$$
$$\big(I(n-1,4), R_1^{(-n+4) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 3$$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$$I(0,4) : -$$
$$I(1,4) : \big(I(0,3), R^1_1(1)\big)$$
$$I(2,4) : \big(I(0,4), R^3_1(2)\big),\ \big(I(1,3), R^3_1(1)\big)$$
$$I(3,4) : \big(I(0,1), R^1_0(1)\big),\ \big(I(0,2), R^2_\infty(1)\big),\ \big(I(1,4), R^2_1(2)\big),\ \big(I(2,3), R^2_1(1)\big),\ \big(2I(0,3), P(2,4)\big)$$
$$I(n,4) : \big(I(n-3,1), R_0^{(-n+3) \bmod 2+1}(1)\big),\ \big(I(n-3,2), R_\infty^{(-n+4) \bmod 2+1}(1)\big),\ \big(I(n-2,4), R_1^{(-n+4) \bmod 3+1}(2)\big)$$
$$\big(I(n-1,3), R_1^{(-n+4) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 3$$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 2$, for $n \geq 0$.

$$I(0,5) : \big(I(0,3), I(2,4)\big),\ \big(I(0,4), I(2,3)\big),\ \big(I(0,6), R^3_1(1)\big)$$

$I(1,5) : \big(I(0,1), I(1,1)\big),\ \big(I(0,2), I(1,2)\big),\ \big(I(1,3), I(3,4)\big),\ \big(I(1,4), I(3,3)\big),\ \big(I(1,6), R_1^2(1)\big)$

$I(n,5) : \big(I(n-1,1), I(n,1)\big),\ \big(I(n-1,2), I(n,2)\big),\ \big(I(n,3), I(n+2,4)\big)$

$\big(I(n,4), I(n+2,3)\big),\ \big(I(n,6), R_1^{(-n+2) \bmod 3+1}(1)\big),\ n > 1$

Modules of the form $I(n,6)$

Defect: $\partial I(n,6) = 2$, for $n \geq 0$.

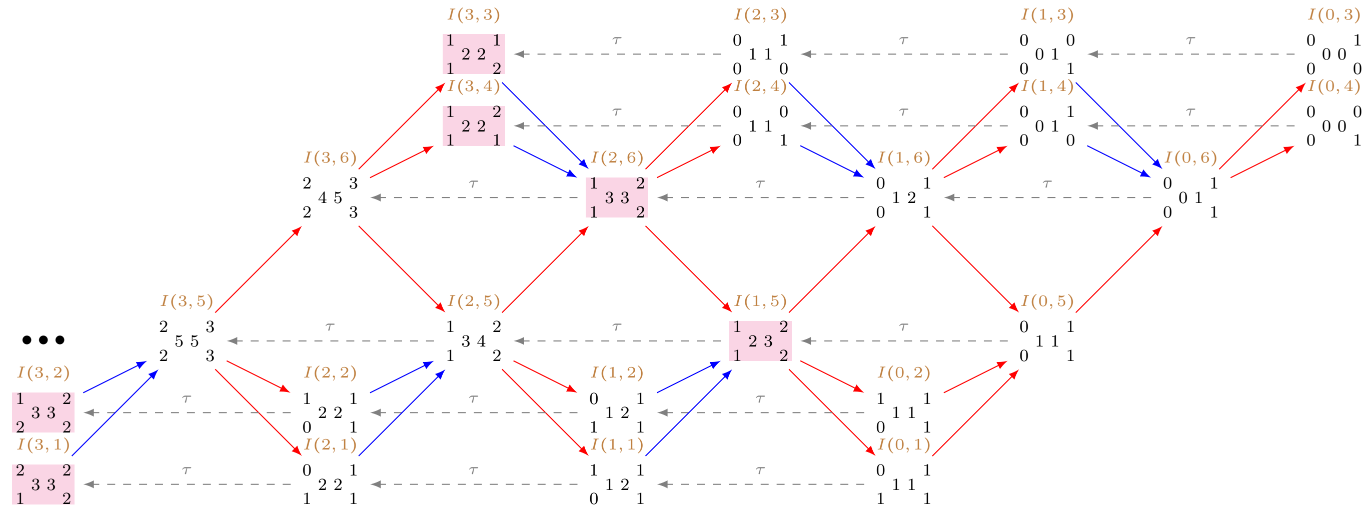
$I(0,6) : \big(I(0,3), I(1,3)\big),\ \big(I(0,4), I(1,4)\big)$

$I(1,6) : \big(I(0,5), R_1^1(1)\big),\ \big(I(1,3), I(2,3)\big),\ \big(I(1,4), I(2,4)\big)$

$I(2,6) : \big(I(0,1), I(2,2)\big),\ \big(I(0,2), I(2,1)\big),\ \big(I(1,5), R_1^3(1)\big),\ \big(I(2,3), I(3,3)\big),\ \big(I(2,4), I(3,4)\big)$

$I(n,6) : \big(I(n-2,1), I(n,2)\big),\ \big(I(n-2,2), I(n,1)\big),\ \big(I(n-1,5), R_1^{(-n+4) \bmod 3+1}(1)\big)$

$\big(I(n,3), I(n+1,3)\big),\ \big(I(n,4), I(n+1,4)\big),\ n > 2$



Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_5)}$

$R_1^1(1) : -$

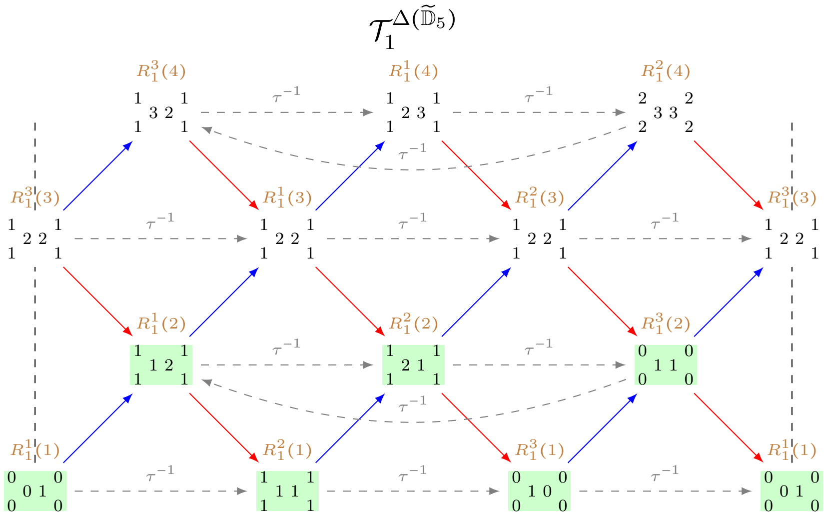
$R_1^1(2) : \big(R_1^2(1), R_1^1(1)\big),\ \big(I(1,1), P(0,1)\big),\ \big(I(1,2), P(0,2)\big),\ \big(I(1,3), P(0,3)\big),\ \big(I(1,4), P(0,4)\big)$

$R_1^2(1) : \big(I(0,2), P(0,1)\big),\ \big(I(0,1), P(0,2)\big),\ \big(I(0,4), P(0,3)\big),\ \big(I(0,3), P(0,4)\big),\ \big(I(0,6), P(0,5)\big)$

$R_1^2(2) : \big(R_1^3(1), R_1^2(1)\big),\ \big(I(0,1), P(1,1)\big),\ \big(I(0,2), P(1,2)\big),\ \big(I(0,3), P(1,3)\big),\ \big(I(0,4), P(1,4)\big)$

$R_1^3(1)$: –

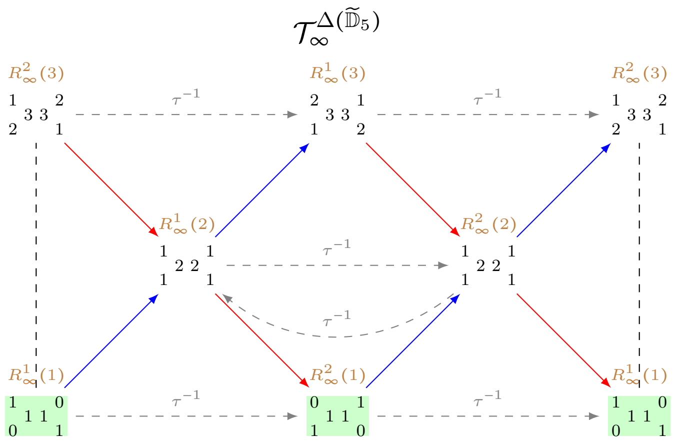
$R_1^3(2) : \big(R_1^1(1), R_1^3(1)\big)$



The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_5)}$

$R_\infty^1(1) : \big(I(2,4), P(0,2)\big),\ \big(I(1,3), P(1,1)\big),\ \big(I(0,4), P(2,2)\big)$

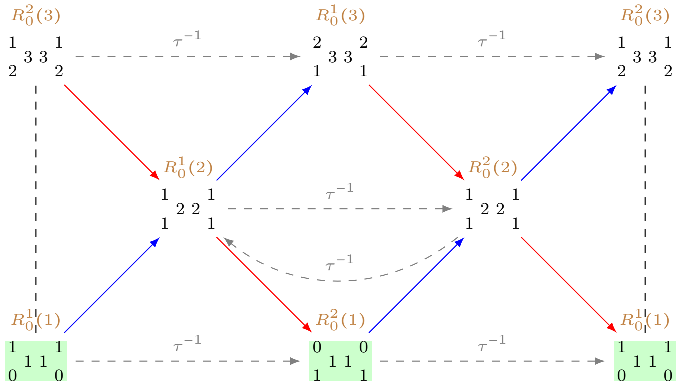
$R_\infty^2(1) : \big(I(2,3), P(0,1)\big),\ \big(I(1,4), P(1,2)\big),\ \big(I(0,3), P(2,1)\big)$



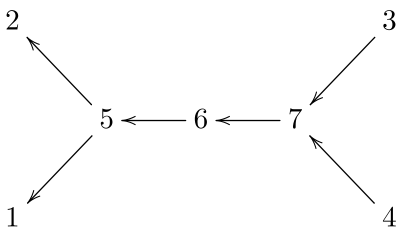
The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_5)}$

$R_0^1(1) : \big(I(2,3), P(0,2)\big),\ \big(I(1,4), P(1,1)\big),\ \big(I(0,3), P(2,2)\big)$
$R_0^2(1) : \big(I(2,4), P(0,1)\big),\ \big(I(1,3), P(1,2)\big),\ \big(I(0,4), P(2,1)\big)$

$\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_5)}$



A.12 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{D}}_6)$ – $\delta = {}^{1}_{1}\,2\,2\,2\,{}^{1}_{1}$



$$C_{\Delta(\widetilde{\mathbb{D}}_6)} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{D}}_6)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1)$: –
$P(1,1) : \big(R_1^3(1), P(0,2)\big)$
$P(2,1) : \big(R_1^3(2), P(0,1)\big),\ \big(R_1^4(1), P(1,2)\big)$
$P(3,1) : \big(R_1^3(3), P(0,2)\big),\ \big(R_1^4(2), P(1,1)\big),\ \big(R_1^1(1), P(2,2)\big)$
$P(4,1) : \big(R_0^2(1), P(0,3)\big),\ \big(R_\infty^2(1), P(0,4)\big),\ \big(R_1^4(3), P(1,2)\big),\ \big(R_1^1(2), P(2,1)\big),\ \big(R_1^2(1), P(3,2)\big)$

$\big(I(3,1), 2P(0,1)\big)$

$P(n,1): \big(R_0^{(n-3) \bmod 2+1}(1), P(n-4,3)\big),\ \big(R_\infty^{(n-3) \bmod 2+1}(1), P(n-4,4)\big),\ \big(R_1^{(n-1) \bmod 4+1}(3), P(n-3,2)\big)$
$\big(R_1^{(n-4) \bmod 4+1}(2), P(n-2,1)\big),\ \big(R_1^{(n-3) \bmod 4+1}(1), P(n-1,2)\big),\ \big(uI,(u+1)P\big),\ n>4$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$.

$P(0,2): -$
$P(1,2): \big(R_1^3(1), P(0,1)\big)$
$P(2,2): \big(R_1^3(2), P(0,2)\big),\ \big(R_1^4(1), P(1,1)\big)$
$P(3,2): \big(R_1^3(3), P(0,1)\big),\ \big(R_1^4(2), P(1,2)\big),\ \big(R_1^1(1), P(2,1)\big)$
$P(4,2): \big(R_\infty^1(1), P(0,3)\big),\ \big(R_0^1(1), P(0,4)\big),\ \big(R_1^4(3), P(1,1)\big),\ \big(R_1^1(2), P(2,2)\big),\ \big(R_1^2(1), P(3,1)\big)$
$\big(I(3,2), 2P(0,2)\big)$
$P(n,2): \big(R_\infty^{(n-4) \bmod 2+1}(1), P(n-4,3)\big),\ \big(R_0^{(n-4) \bmod 2+1}(1), P(n-4,4)\big),\ \big(R_1^{(n-1) \bmod 4+1}(3), P(n-3,1)\big)$
$\big(R_1^{(n-4) \bmod 4+1}(2), P(n-2,2)\big),\ \big(R_1^{(n-3) \bmod 4+1}(1), P(n-1,1)\big),\ \big(uI,(u+1)P\big),\ n>4$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -1$, for $n \geq 0$.

$P(0,3): \big(R_0^1(1), P(0,1)\big),\ \big(R_\infty^2(1), P(0,2)\big),\ \big(I(2,3), P(0,5)\big),\ \big(I(1,4), P(0,6)\big),\ \big(I(0,3), P(0,7)\big)$
$P(1,3): \big(R_1^3(1), P(0,4)\big),\ \big(R_0^2(1), P(1,1)\big),\ \big(R_\infty^1(1), P(1,2)\big),\ \big(I(1,3), P(1,5)\big),\ \big(I(0,4), P(1,6)\big)$
$P(2,3): \big(R_1^3(2), P(0,3)\big),\ \big(R_1^4(1), P(1,4)\big),\ \big(R_0^1(1), P(2,1)\big),\ \big(R_\infty^2(1), P(2,2)\big),\ \big(I(0,3), P(2,5)\big)$
$P(3,3): \big(R_1^3(3), P(0,4)\big),\ \big(R_1^4(2), P(1,3)\big),\ \big(R_1^1(1), P(2,4)\big),\ \big(R_0^2(1), P(3,1)\big),\ \big(R_\infty^1(1), P(3,2)\big)$
$P(4,3): \big(R_1^4(3), P(1,4)\big),\ \big(R_1^1(2), P(2,3)\big),\ \big(R_1^2(1), P(3,4)\big),\ \big(R_0^1(1), P(4,1)\big),\ \big(R_\infty^2(1), P(4,2)\big)$
$\big(I(3,3), 2P(0,3)\big)$
$P(n,3): \big(R_1^{(n-1) \bmod 4+1}(3), P(n-3,4)\big),\ \big(R_1^{(n-4) \bmod 4+1}(2), P(n-2,3)\big),\ \big(R_1^{(n-3) \bmod 4+1}(1), P(n-1,4)\big)$
$\big(R_0^{(n-4) \bmod 2+1}(1), P(n,1)\big),\ \big(R_\infty^{(n-3) \bmod 2+1}(1), P(n,2)\big),\ \big(uI,(u+1)P\big),\ n>4$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4) = -1$, for $n \geq 0$.

$P(0,4): \big(R_\infty^1(1), P(0,1)\big),\ \big(R_0^2(1), P(0,2)\big),\ \big(I(2,4), P(0,5)\big),\ \big(I(1,3), P(0,6)\big),\ \big(I(0,4), P(0,7)\big)$
$P(1,4): \big(R_1^3(1), P(0,3)\big),\ \big(R_\infty^2(1), P(1,1)\big),\ \big(R_0^1(1), P(1,2)\big),\ \big(I(1,4), P(1,5)\big),\ \big(I(0,3), P(1,6)\big)$
$P(2,4): \big(R_1^3(2), P(0,4)\big),\ \big(R_1^4(1), P(1,3)\big),\ \big(R_\infty^1(1), P(2,1)\big),\ \big(R_0^2(1), P(2,2)\big),\ \big(I(0,4), P(2,5)\big)$

$P(3,4):\big(R_1^3(3),P(0,3)\big),\ \big(R_1^4(2),P(1,4)\big),\ \big(R_1^1(1),P(2,3)\big),\ \big(R_\infty^2(1),P(3,1)\big),\ \big(R_0^1(1),P(3,2)\big)$
$P(4,4):\big(R_1^4(3),P(1,3)\big),\ \big(R_1^1(2),P(2,4)\big),\ \big(R_1^2(1),P(3,3)\big),\ \big(R_\infty^1(1),P(4,1)\big),\ \big(R_0^2(1),P(4,2)\big)$
$\big(I(3,4),2P(0,4)\big)$
$P(n,4):\big(R_1^{(n-1)\bmod 4+1}(3),P(n-3,3)\big),\ \big(R_1^{(n-4)\bmod 4+1}(2),P(n-2,4)\big),\ \big(R_1^{(n-3)\bmod 4+1}(1),P(n-1,3)\big)$
$\big(R_\infty^{(n-4)\bmod 2+1}(1),P(n,1)\big),\ \big(R_0^{(n-3)\bmod 2+1}(1),P(n,2)\big),\ \big(uI,(u+1)P\big),\ n>4$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5)=-2$, for $n\geq 0$.

$P(0,5):\big(P(1,1),P(0,1)\big),\ \big(P(1,2),P(0,2)\big)$
$P(1,5):\big(R_1^3(1),P(0,6)\big),\ \big(P(2,1),P(1,1)\big),\ \big(P(2,2),P(1,2)\big)$
$P(2,5):\big(R_1^3(2),P(0,7)\big),\ \big(R_1^4(1),P(1,6)\big),\ \big(P(3,1),P(2,1)\big),\ \big(P(3,2),P(2,2)\big)$
$P(3,5):\big(P(3,3),P(0,3)\big),\ \big(P(3,4),P(0,4)\big),\ \big(R_1^4(2),P(1,7)\big),\ \big(R_1^1(1),P(2,6)\big),\ \big(P(4,1),P(3,1)\big)$
$\big(P(4,2),P(3,2)\big)$
$P(n,5):\big(P(n,3),P(n-3,3)\big),\ \big(P(n,4),P(n-3,4)\big),\ \big(R_1^{n\bmod 4+1}(2),P(n-2,7)\big)$
$\big(R_1^{(n-3)\bmod 4+1}(1),P(n-1,6)\big),\ \big(P(n+1,1),P(n,1)\big),\ \big(P(n+1,2),P(n,2)\big),\ n>3$

Modules of the form $P(n,6)$

Defect: $\partial P(n,6)=-2$, for $n\geq 0$.

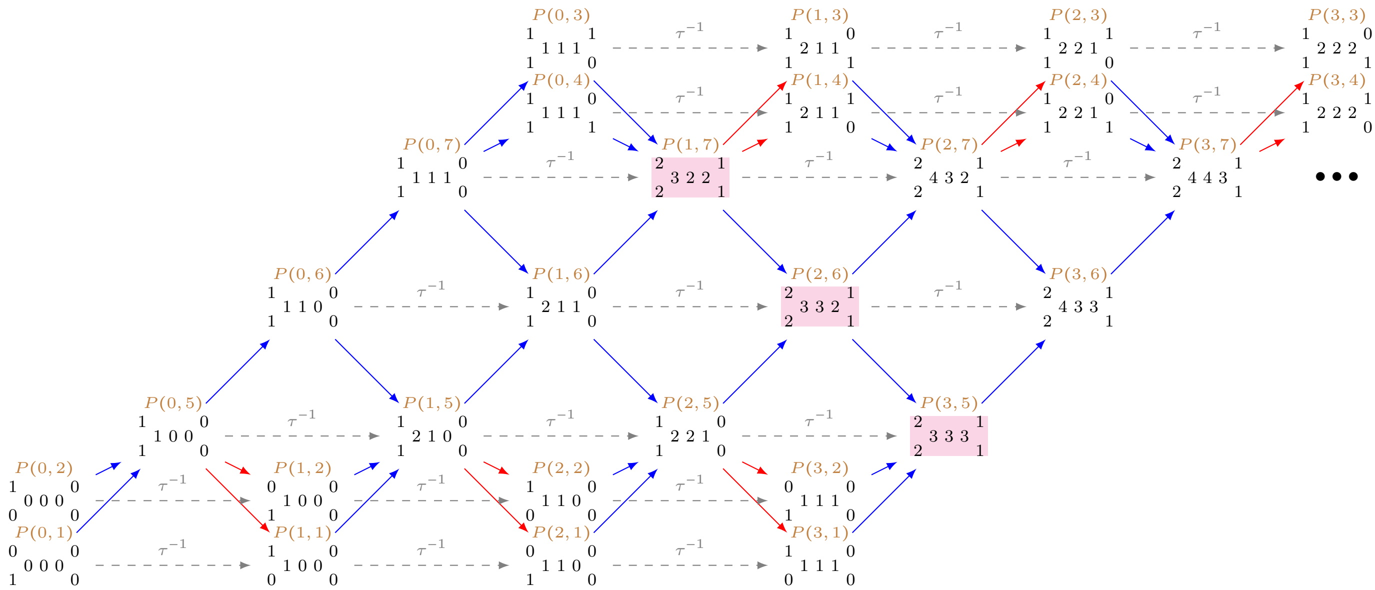
$P(0,6):\big(P(2,2),P(0,1)\big),\ \big(P(2,1),P(0,2)\big),\ \big(R_1^4(1),P(0,5)\big)$
$P(1,6):\big(R_1^3(1),P(0,7)\big),\ \big(P(3,2),P(1,1)\big),\ \big(P(3,1),P(1,2)\big),\ \big(R_1^1(1),P(1,5)\big)$
$P(2,6):\big(P(2,4),P(0,3)\big),\ \big(P(2,3),P(0,4)\big),\ \big(R_1^4(1),P(1,7)\big),\ \big(P(4,2),P(2,1)\big),\ \big(P(4,1),P(2,2)\big)$
$\big(R_1^2(1),P(2,5)\big)$
$P(n,6):\big(P(n,4),P(n-2,3)\big),\ \big(P(n,3),P(n-2,4)\big),\ \big(R_1^{(n+1)\bmod 4+1}(1),P(n-1,7)\big)$
$\big(P(n+2,2),P(n,1)\big),\ \big(P(n+2,1),P(n,2)\big),\ \big(R_1^{(n-1)\bmod 4+1}(1),P(n,5)\big),\ n>2$

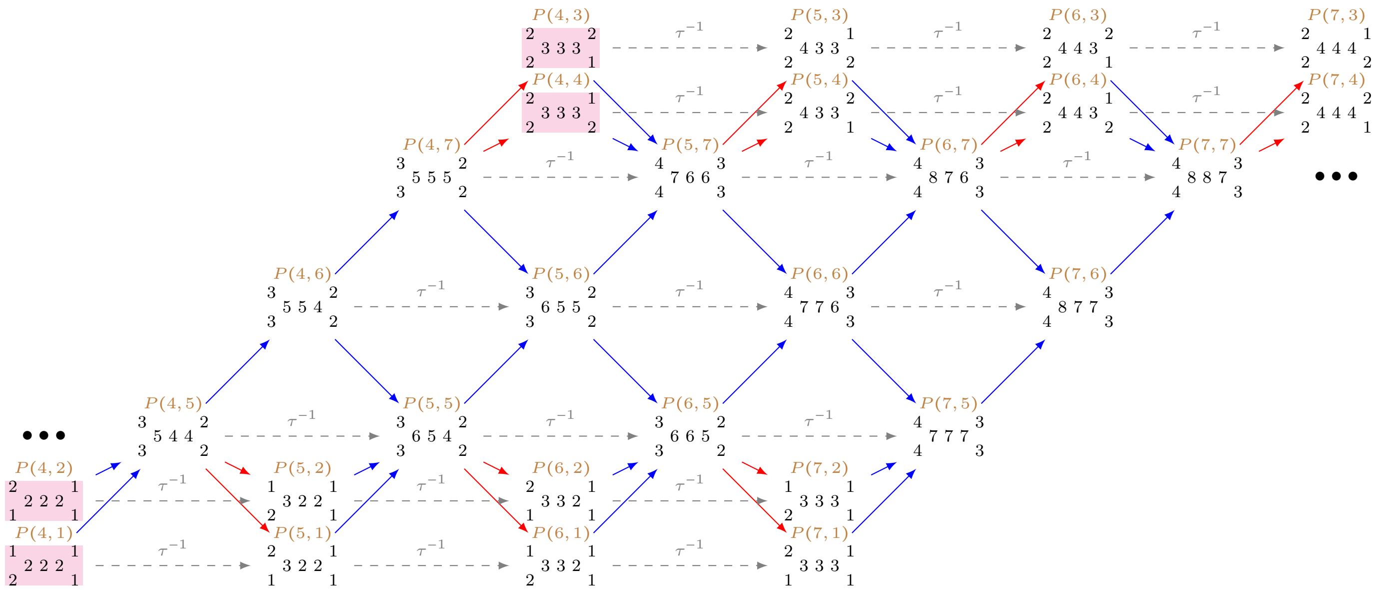
Modules of the form $P(n,7)$

Defect: $\partial P(n,7)=-2$, for $n\geq 0$.

$P(0,7):\big(P(3,1),P(0,1)\big),\ \big(P(3,2),P(0,2)\big),\ \big(R_1^4(2),P(0,5)\big),\ \big(R_1^1(1),P(0,6)\big)$
$P(1,7):\big(P(1,3),P(0,3)\big),\ \big(P(1,4),P(0,4)\big),\ \big(P(4,1),P(1,1)\big),\ \big(P(4,2),P(1,2)\big),\ \big(R_1^1(2),P(1,5)\big)$
$\big(R_1^2(1),P(1,6)\big)$
$P(n,7):\big(P(n,3),P(n-1,3)\big),\ \big(P(n,4),P(n-1,4)\big),\ \big(P(n+3,1),P(n,1)\big)$

$$\big(P(n+3,2),P(n,2)\big),\ \big(R_1^{(n-1)\bmod 4+1}(2),P(n,5)\big),\ \big(R_1^{n\bmod 4+1}(1),P(n,6)\big),\ n>1$$





Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1)=1$, for $n\geq 0$.

$I(0,1): \big(I(0,3), R_0^2(1)\big),\ \big(I(0,4), R_\infty^2(1)\big),\ \big(I(0,5), P(0,1)\big),\ \big(I(0,6), P(1,2)\big),\ \big(I(0,7), P(2,1)\big)$

$I(1,1): \big(I(0,2), R_1^1(1)\big),\ \big(I(1,3), R_0^1(1)\big),\ \big(I(1,4), R_\infty^1(1)\big),\ \big(I(1,6), P(0,2)\big),\ \big(I(1,7), P(1,1)\big)$

$I(2,1): \big(I(0,1), R_1^4(2)\big),\ \big(I(1,2), R_1^4(1)\big),\ \big(I(2,3), R_0^2(1)\big),\ \big(I(2,4), R_\infty^2(1)\big),\ \big(I(2,7), P(0,1)\big)$

$I(3,1): \big(I(0,2), R_1^3(3)\big),\ \big(I(1,1), R_1^3(2)\big),\ \big(I(2,2), R_1^3(1)\big),\ \big(I(3,3), R_0^1(1)\big),\ \big(I(3,4), R_\infty^1(1)\big)$

$I(4,1): \big(I(1,2), R_1^2(3)\big),\ \big(I(2,1), R_1^2(2)\big),\ \big(I(3,2), R_1^2(1)\big),\ \big(I(4,3), R_0^2(1)\big),\ \big(I(4,4), R_\infty^2(1)\big)$
$\big(2I(0,1), P(3,1)\big)$

$I(n,1): \big(I(n-3,2), R_1^{(-n+5) \bmod 4+1}(3)\big),\ \big(I(n-2,1), R_1^{(-n+5) \bmod 4+1}(2)\big),\ \big(I(n-1,2), R_1^{(-n+5) \bmod 4+1}(1)\big)$
$\big(I(n,3), R_0^{(-n+5) \bmod 2+1}(1)\big),\ \big(I(n,4), R_\infty^{(-n+5) \bmod 2+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 4$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$.

$I(0,2): \big(I(0,3), R_\infty^1(1)\big),\ \big(I(0,4), R_0^1(1)\big),\ \big(I(0,5), P(0,2)\big),\ \big(I(0,6), P(1,1)\big),\ \big(I(0,7), P(2,2)\big)$

$I(1,2): \big(I(0,1), R_1^1(1)\big),\ \big(I(1,3), R_\infty^2(1)\big),\ \big(I(1,4), R_0^2(1)\big),\ \big(I(1,6), P(0,1)\big),\ \big(I(1,7), P(1,2)\big)$

$I(2,2): \big(I(0,2), R_1^4(2)\big),\ \big(I(1,1), R_1^4(1)\big),\ \big(I(2,3), R_\infty^1(1)\big),\ \big(I(2,4), R_0^1(1)\big),\ \big(I(2,7), P(0,2)\big)$

$I(3,2): \big(I(0,1), R_1^3(3)\big),\ \big(I(1,2), R_1^3(2)\big),\ \big(I(2,1), R_1^3(1)\big),\ \big(I(3,3), R_\infty^2(1)\big),\ \big(I(3,4), R_0^2(1)\big)$

$I(4,2): \big(I(1,1), R_1^2(3)\big),\ \big(I(2,2), R_1^2(2)\big),\ \big(I(3,1), R_1^2(1)\big),\ \big(I(4,3), R_\infty^1(1)\big),\ \big(I(4,4), R_0^1(1)\big)$
$\big(2I(0,2), P(3,2)\big)$

$I(n,2): \big(I(n-3,1), R_1^{(-n+5) \bmod 4+1}(3)\big),\ \big(I(n-2,2), R_1^{(-n+5) \bmod 4+1}(2)\big),\ \big(I(n-1,1), R_1^{(-n+5) \bmod 4+1}(1)\big)$
$\big(I(n,3), R_\infty^{(-n+4) \bmod 2+1}(1)\big),\ \big(I(n,4), R_0^{(-n+4) \bmod 2+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 4$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$.

$I(0,3): -$

$I(1,3): \big(I(0,4), R_1^1(1)\big)$

$I(2,3): \big(I(0,3), R_1^4(2)\big),\ \big(I(1,4), R_1^4(1)\big)$

$I(3,3): \big(I(0,4), R_1^3(3)\big),\ \big(I(1,3), R_1^3(2)\big),\ \big(I(2,4), R_1^3(1)\big)$

$I(4,3): \big(I(0,1), R_0^1(1)\big),\ \big(I(0,2), R_\infty^2(1)\big),\ \big(I(1,4), R_1^2(3)\big),\ \big(I(2,3), R_1^2(2)\big),\ \big(I(3,4), R_1^2(1)\big)$
$\big(2I(0,3), P(3,3)\big)$

$I(n,3): \big(I(n-4,1), R_0^{(-n+4) \bmod 2+1}(1)\big),\ \big(I(n-4,2), R_\infty^{(-n+5) \bmod 2+1}(1)\big),\ \big(I(n-3,4), R_1^{(-n+5) \bmod 4+1}(3)\big)$
$\big(I(n-2,3), R_1^{(-n+5) \bmod 4+1}(2)\big),\ \big(I(n-1,4), R_1^{(-n+5) \bmod 4+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 4$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 1$, for $n \geq 0$.

$I(0,4):-$
$I(1,4):\big(I(0,3),R_1^1(1)\big)$
$I(2,4):\big(I(0,4),R_1^4(2)\big),\ \big(I(1,3),R_1^4(1)\big)$
$I(3,4):\big(I(0,3),R_1^3(3)\big),\ \big(I(1,4),R_1^3(2)\big),\ \big(I(2,3),R_1^3(1)\big)$
$I(4,4):\big(I(0,1),R_\infty^1(1)\big),\ \big(I(0,2),R_0^2(1)\big),\ \big(I(1,3),R_1^2(3)\big),\ \big(I(2,4),R_1^2(2)\big),\ \big(I(3,3),R_1^2(1)\big)$
$\big(2I(0,4),P(3,4)\big)$
$I(n,4):\big(I(n-4,1),R_\infty^{(-n+4)\text{ mod }2+1}(1)\big),\ \big(I(n-4,2),R_0^{(-n+5)\text{ mod }2+1}(1)\big),\ \big(I(n-3,3),R_1^{(-n+5)\text{ mod }4+1}(3)\big)$
$\big(I(n-2,4),R_1^{(-n+5)\text{ mod }4+1}(2)\big),\ \big(I(n-1,3),R_1^{(-n+5)\text{ mod }4+1}(1)\big),\ \big((v+1)I,vP\big),\ n>4$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 2$, for $n \geq 0$.

$I(0,5):\big(I(0,3),I(3,3)\big),\ \big(I(0,4),I(3,4)\big),\ \big(I(0,6),R_1^3(1)\big),\ \big(I(0,7),R_1^3(2)\big)$
$I(1,5):\big(I(0,1),I(1,1)\big),\ \big(I(0,2),I(1,2)\big),\ \big(I(1,3),I(4,3)\big),\ \big(I(1,4),I(4,4)\big),\ \big(I(1,6),R_1^2(1)\big)$
$\big(I(1,7),R_1^2(2)\big)$
$I(n,5):\big(I(n-1,1),I(n,1)\big),\ \big(I(n-1,2),I(n,2)\big),\ \big(I(n,3),I(n+3,3)\big)$
$\big(I(n,4),I(n+3,4)\big),\ \big(I(n,6),R_1^{(-n+2)\text{ mod }4+1}(1)\big),\ \big(I(n,7),R_1^{(-n+2)\text{ mod }4+1}(2)\big),\ n>1$

Modules of the form $I(n,6)$

Defect: $\partial I(n,6) = 2$, for $n \geq 0$.

$I(0,6):\big(I(0,3),I(2,4)\big),\ \big(I(0,4),I(2,3)\big),\ \big(I(0,7),R_1^4(1)\big)$
$I(1,6):\big(I(0,5),R_1^1(1)\big),\ \big(I(1,3),I(3,4)\big),\ \big(I(1,4),I(3,3)\big),\ \big(I(1,7),R_1^3(1)\big)$
$I(2,6):\big(I(0,1),I(2,2)\big),\ \big(I(0,2),I(2,1)\big),\ \big(I(1,5),R_1^4(1)\big),\ \big(I(2,3),I(4,4)\big),\ \big(I(2,4),I(4,3)\big)$
$\big(I(2,7),R_1^2(1)\big)$
$I(n,6):\big(I(n-2,1),I(n,2)\big),\ \big(I(n-2,2),I(n,1)\big),\ \big(I(n-1,5),R_1^{(-n+5)\text{ mod }4+1}(1)\big)$
$\big(I(n,3),I(n+2,4)\big),\ \big(I(n,4),I(n+2,3)\big),\ \big(I(n,7),R_1^{(-n+3)\text{ mod }4+1}(1)\big),\ n>2$

Modules of the form $I(n,7)$

Defect: $\partial I(n,7) = 2$, for $n \geq 0$.

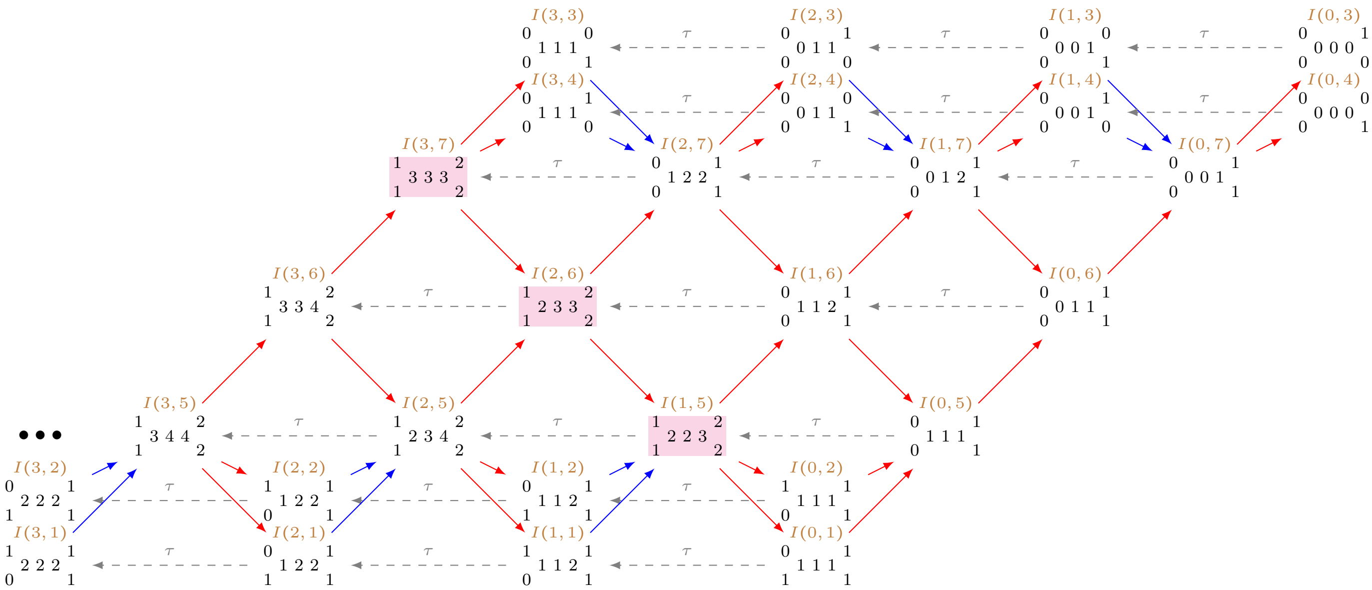
$I(0,7): \big(I(0,3), I(1,3)\big),\ \big(I(0,4), I(1,4)\big)$

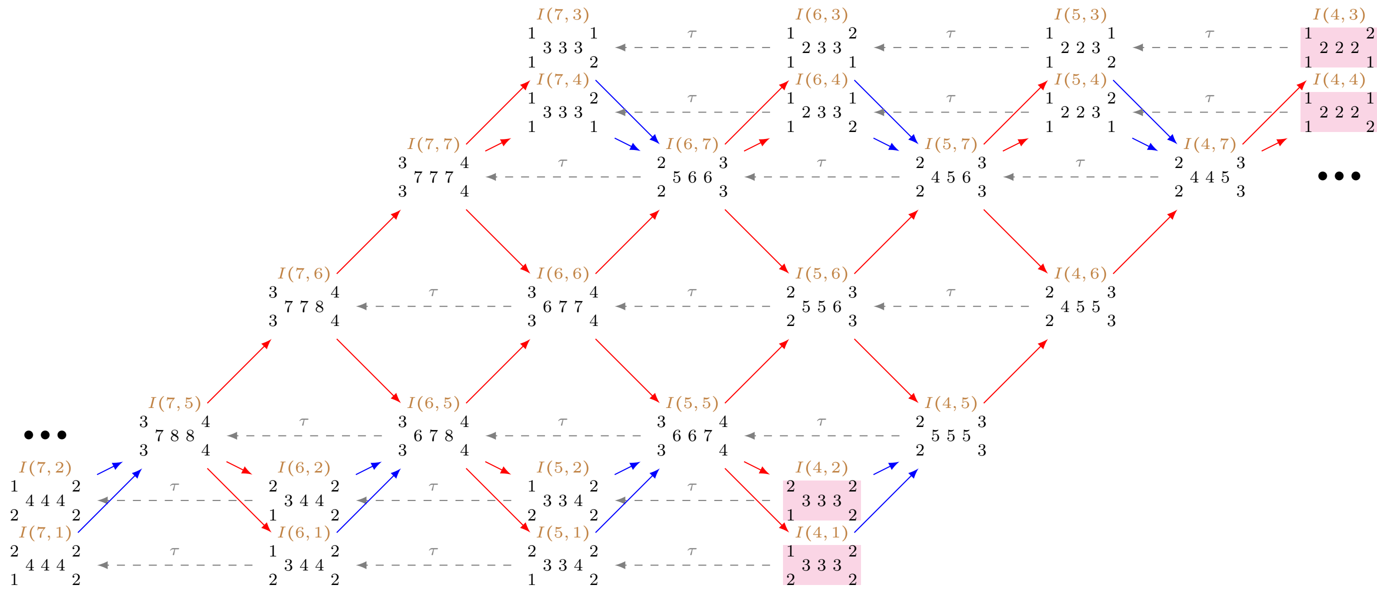
$I(1,7): \big(I(0,6), R_1^1(1)\big),\ \big(I(1,3), I(2,3)\big),\ \big(I(1,4), I(2,4)\big)$

$I(2,7): \big(I(0,5), R_1^4(2)\big),\ \big(I(1,6), R_1^4(1)\big),\ \big(I(2,3), I(3,3)\big),\ \big(I(2,4), I(3,4)\big)$

$I(3,7): \big(I(0,1), I(3,1)\big),\ \big(I(0,2), I(3,2)\big),\ \big(I(1,5), R_1^3(2)\big),\ \big(I(2,6), R_1^3(1)\big),\ \big(I(3,3), I(4,3)\big)$
$\big(I(3,4), I(4,4)\big)$

$I(n,7): \big(I(n-3,1), I(n,1)\big),\ \big(I(n-3,2), I(n,2)\big),\ \big(I(n-2,5), R_1^{(-n+5) \bmod 4+1}(2)\big)$
$\big(I(n-1,6), R_1^{(-n+5) \bmod 4+1}(1)\big),\ \big(I(n,3), I(n+1,3)\big),\ \big(I(n,4), I(n+1,4)\big),\ n > 3$





Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_6)}$

$R_1^1(1)$: –

$R_1^1(2)$: $(R_1^2(1), R_1^1(1))$, $(I(1,1), P(0,1))$, $(I(1,2), P(0,2))$, $(I(1,3), P(0,3))$, $(I(1,4), P(0,4))$ $(I(1,7), P(0,5))$

$R_1^2(1)$: $(I(0,2), P(0,1))$, $(I(0,1), P(0,2))$, $(I(0,4), P(0,3))$, $(I(0,3), P(0,4))$, $(I(0,6), P(0,5))$ $(I(0,7), P(0,6))$

$R_1^2(2)$: $(R_1^3(1), R_1^2(1))$, $(I(0,1), P(1,1))$, $(I(0,2), P(1,2))$, $(I(0,3), P(1,3))$, $(I(0,4), P(1,4))$ $(I(0,7), P(1,5))$

$R_1^3(1)$: –

$R_1^3(2)$: $(R_1^4(1), R_1^3(1))$

$R_1^4(1)$: –

$R_1^4(2)$: $(R_1^1(1), R_1^4(1))$

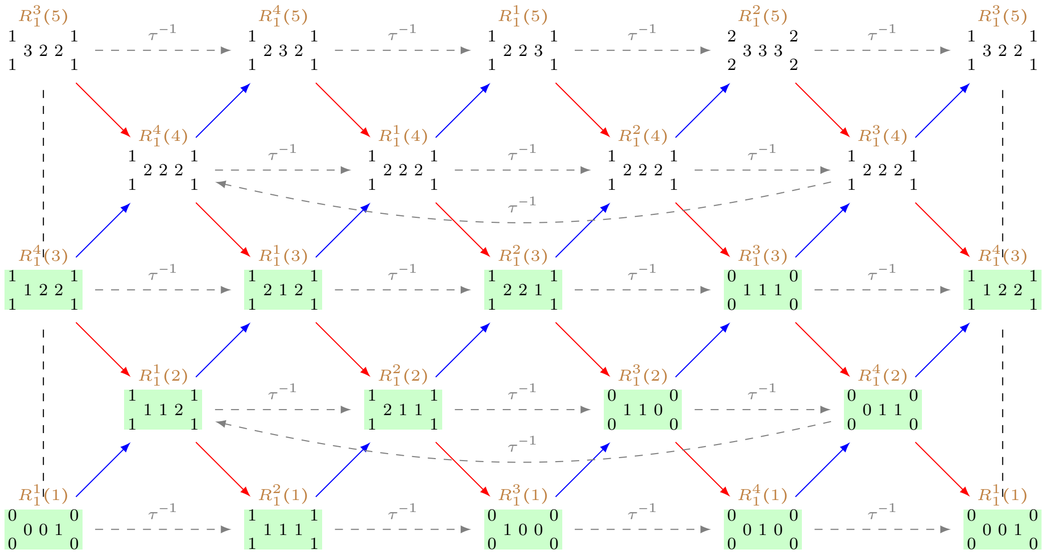
$R_1^4(3)$: $(R_1^1(2), R_1^4(1))$, $(R_1^2(1), R_1^4(2))$, $(I(2,2), P(0,1))$, $(I(2,1), P(0,2))$, $(I(2,4), P(0,3))$ $(I(2,3), P(0,4))$

$R_1^1(3)$: $(R_1^2(2), R_1^1(1))$, $(R_1^3(1), R_1^1(2))$, $(I(1,2), P(1,1))$, $(I(1,1), P(1,2))$, $(I(1,4), P(1,3))$ $(I(1,3), P(1,4))$

$R_1^2(3)$: $(R_1^3(2), R_1^2(1))$, $(R_1^4(1), R_1^2(2))$, $(I(0,2), P(2,1))$, $(I(0,1), P(2,2))$, $(I(0,4), P(2,3))$ $(I(0,3), P(2,4))$

$R_1^3(3)$: $(R_1^4(2), R_1^3(1))$, $(R_1^1(1), R_1^3(2))$

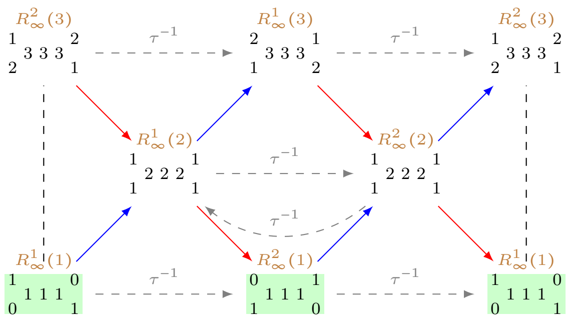
$\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_6)}$



The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_6)}$

$R_\infty^1(1) : \big(I(3,3), P(0,2)\big),\ \big(I(2,4), P(1,1)\big),\ \big(I(1,3), P(2,2)\big),\ \big(I(0,4), P(3,1)\big)$
$R_\infty^2(1) : \big(I(3,4), P(0,1)\big),\ \big(I(2,3), P(1,2)\big),\ \big(I(1,4), P(2,1)\big),\ \big(I(0,3), P(3,2)\big)$

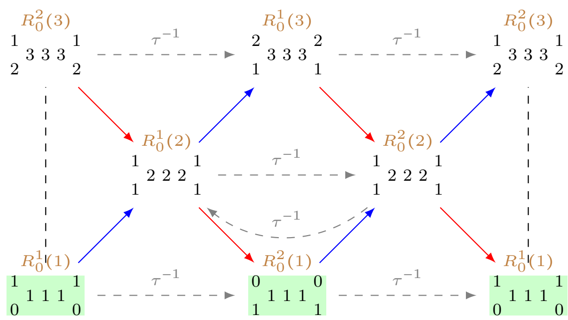
$\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_6)}$



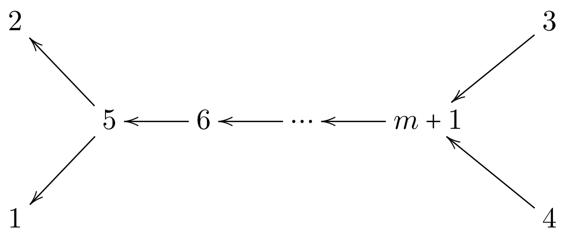
The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_6)}$

$R_0^1(1) : \big(I(3,4), P(0,2)\big),\ \big(I(2,3), P(1,1)\big),\ \big(I(1,4), P(2,2)\big),\ \big(I(0,3), P(3,1)\big)$
$R_0^2(1) : \big(I(3,3), P(0,1)\big),\ \big(I(2,4), P(1,2)\big),\ \big(I(1,3), P(2,1)\big),\ \big(I(0,4), P(3,2)\big)$

$\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_6)}$



A.13 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{D}}_m)$ – $\delta = {}^{1}_{1}\,2\,2\cdots 2\,{}^{1}_{1}$, for $m \geq 4$



$$C_{\Delta(\widetilde{\mathbb{D}}_m)} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \qquad \Phi_{\Delta(\widetilde{\mathbb{D}}_m)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & \cdots & 1 \\ -1 & -1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

$\underline{\dim}P(0,j) = j^{th}$ column of $C_{\Delta(\widetilde{\mathbb{D}}_m)}$, $\underline{\dim}I(0,i) = i^{th}$ row of $C_{\Delta(\widetilde{\mathbb{D}}_m)}$

Throughout the lists below, in Schofield pairs of the form (R, P) associated to the preprojective indecomposable P', the preprojective indecomposable P is given by an explicit formula, while R always denotes the non-homogeneous regular with dimension vector $\underline{\dim}R = \underline{\dim}P' - \underline{\dim}P$ (for each such pair separately).

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$. In the formulas below $0 \leq n_1 < m-2$ and $n_2 > m-2$:

$$\begin{aligned}
P(n_1,1) &: \big(R, P(i,(n_1+i) \bmod 2+1)\big),\ 0 \leq i \leq n_1-1 \\
P(m-2,1) &: \big(R, P(i,(m+i) \bmod 2+1)\big),\ 1 \leq i \leq m-3 \\
&\quad \big(R, P(0,3)\big),\ \big(R, P(0,4)\big),\ \big(I(m-3,1), 2P(0, m \bmod 2+1)\big) \\
P(n_2,1) &: \big(R, P(n_2-m+i+2,(m+i) \bmod 2+1)\big),\ 1 \leq i \leq m-3 \\
&\quad \big(R, P(n_2-m+2,3)\big),\ \big(R, P(n_2-m+2,4)\big),\ \big(uI,(u+1)P\big)
\end{aligned}$$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -1$, for $n \geq 0$. In the formulas below $0 \leq n_1 < m-2$ and $n_2 > m-2$:

$$\begin{aligned}
P(n_1,2) &: \big(R, P(i,(n_1+i+1) \bmod 2+1)\big),\ 0 \leq i \leq n_1-1 \\
P(m-2,2) &: \big(R, P(i,(m+i+1) \bmod 2+1)\big),\ 1 \leq i \leq m-3 \\
&\quad \big(R, P(0,3)\big),\ \big(R, P(0,4)\big),\ \big(I(m-3,2), 2P(0,(m+1) \bmod 2+1)\big) \\
P(n_2,2) &: \big(R, P(n_2-m+i+2,(m+i+1) \bmod 2+1)\big),\ 1 \leq i \leq m-3 \\
&\quad \big(R, P(n_2-m+2,3)\big),\ \big(R, P(n_2-m+2,4)\big),\ \big(uI,(u+1)P\big)
\end{aligned}$$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -1$, for $n \geq 0$. In the formulas below $0 \leq n_1 < m-2$ and $n_2 > m-2$:

$$\begin{aligned}
P(n_1,3) &: \big(R, P(i,(n_1+i) \bmod 2+3)\big),\ 0 \leq i \leq n_1-1 \\
&\quad \big(I(m-n_1-i+1,(m+i+1) \bmod 2+3), P(n_1,i)\big),\ 5 \leq i \leq m-n_1+1 \\
&\quad \big(R, P(n_1,1)\big),\ \big(R, P(n_1,2)\big) \\
P(m-2,3) &: \big(R, P(i,(m+i) \bmod 2+3)\big),\ 1 \leq i \leq m-3 \\
&\quad \big(R, P(m-2,1)\big),\ \big(R, P(m-2,2)\big),\ \big(I(m-3,3), 2P(0, m \bmod 2+3)\big) \\
P(n_2,3) &: \big(R, P(n_2-m+i+2,(m+i) \bmod 2+3)\big),\ 1 \leq i \leq m-3 \\
&\quad \big(R, P(n_2,1)\big),\ \big(R, P(n_2,2)\big),\ \big(uI,(u+1)P\big)
\end{aligned}$$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4) = -1$, for $n \geq 0$. In the formulas below $0 \leq n_1 < m-2$ and $n_2 > m-2$:

$$\begin{aligned}
P(n_1,4) &: \big(R, P(i,(n_1+i+1) \bmod 2+3)\big),\ 0 \leq i \leq n_1-1 \\
&\quad \big(I(m-n_1-i+1,(m+i) \bmod 2+3), P(n_1,i)\big),\ 5 \leq i \leq m-n_1+1 \\
&\quad \big(R, P(n_1,1)\big),\ \big(R, P(n_1,2)\big) \\
P(m-2,4) &: \big(R, P(i,(m+i+1) \bmod 2+3)\big),\ 1 \leq i \leq m-3 \\
&\quad \big(R, P(m-2,1)\big),\ \big(R, P(m-2,2)\big),\ \big(I(m-3,4), 2P(0,(m+1) \bmod 2+3)\big) \\
P(n_2,4) &: \big(R, P(n_2-m+i+2,(m+i+1) \bmod 2+3)\big),\ 1 \leq i \leq m-3
\end{aligned}$$

$$\big(R, P(n_2,1)\big),\ \big(R, P(n_2,2)\big),\ \big(uI, (u+1)P\big)$$

Modules of the form $P(n,j)$, for $5 \le j \le m+1$

Defect: $\partial P(n,j) = -2$, for $n \ge 0$. In the formulas below $0 \le n_1 < m-j+1$ and $n_2 > m-j+1$:

$$\begin{aligned}
P(n_1,j):\ & \big(R, P(i, j+n_1-i)\big),\ 0 \le i \le n_1-1\\
& \big(R, P(n_1,i)\big),\ 5 \le i \le j-1\\
& \big(P(j-4+n_1, (j+1) \bmod 2+1), P(n_1,1)\big),\ \big(P(j-4+n_1, j \bmod 2+1), P(n_1,2)\big)\\
P(m-j+2,j):\ & \big(R, P(i+1, m-i+1)\big),\ 0 \le i \le m-j\\
& \big(R, P(m-j+2,i)\big),\ 5 \le i \le j-1\\
& \big(P(m-j+2, (m+j+1) \bmod 2+3), P(0,3)\big),\ \big(P(m-j+2, 4-(m+j+1) \bmod 2), P(0,4)\big),\\
& \big(P(m-2, (j+1) \bmod 2+1), P(m-j+2,1)\big),\ \big(P(m-2, j \bmod 2+1), P(m-j+2,2)\big)\\
P(n_2,j):\ & \big(R, P(n_2-m+j+i, m-i+1)\big),\ 0 \le i \le m-j\\
& \big(R, P(n_2+1,i)\big),\ 5 \le i \le j-1\\
& \big(P(n_2+1, (m+j+1) \bmod 2+3), P(n_2-m+j-1,3)\big),\\
& \big(P(n_2+1, 4-(m+j+1) \bmod 2), P(n_2-m+j-1,4)\big),\\
& \big(P(n_2+j-3, (j+1) \bmod 2+1), P(n_2+1,1)\big),\\
& \big(P(n_2+j-3, j \bmod 2+1), P(n_2+1,2)\big)
\end{aligned}$$

Schofield pairs associated to preinjective exceptional modules

$\underline{\dim}P(0,j) = j^{th}$ column of $C_{\Delta(\widetilde{\mathbb{D}}_m)}$, $\underline{\dim}I(0,i) = i^{th}$ row of $C_{\Delta(\widetilde{\mathbb{D}}_m)}$

Throughout the lists below, in Schofield pairs of the form (I,R) associated to the preinjective indecomposable I', the preinjective indecomposable I is given by an explicit formula, while R always denotes the non-homogeneous regular with dimension vector $\underline{\dim}R = \underline{\dim}I' - \underline{\dim}I$ (for each such pair separately).

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \ge 0$. In the formulas below $0 \le n_1 < m-2$ and $n_2 > m-2$:

$$\begin{aligned}
I(n_1,1):\ & \big(I(i, (n_1+i) \bmod 2+1), R\big),\ 0 \le i \le n_1-1\\
& \big(I(n_1,i), P(i-n_1-5, (i+1) \bmod 2+1)\big),\ 5+n_1 \le i \le m+1\\
& \big(I(n_1,3), R\big),\ \big(I(n_1,4), R\big)\\
I(m-2,1):\ & \big(I(i, (m+i) \bmod 2+1), R\big),\ 1 \le i \le m-3\\
& \big(I(m-2,3), R\big),\ \big(I(m-2,4), R\big),\ \big(2I(0, m \bmod 2+1), P(m-3,1)\big)\\
I(n_2,1):\ & \big(I(n_2-m+i+2, (m+i) \bmod 2+1), R\big),\ 1 \le i \le m-3\\
& \big(I(n_2,3), R\big),\ \big(I(n_2,4), R\big),\ \big((v+1)I, vP\big)
\end{aligned}$$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 1$, for $n \geq 0$. In the formulas below $0 \leq n_1 < m-2$ and $n_2 > m-2$:

$$\begin{aligned}
I(n_1,2):&\ \big(I(i,(n_1+i+1) \bmod 2+1), R\big),\ 0 \leq i \leq n_1-1\\
&\ \big(I(n_1,i), P(i-n_1-5, i \bmod 2+1)\big),\ 5+n_1 \leq i \leq m+1\\
&\ \big(I(n_1,3), R\big),\ \big(I(n_1,4)\big)\\
I(m-2,2):&\ \big(I(i,(m+i+1) \bmod 2+1), R\big),\ 1 \leq i \leq m-3\\
&\ \big(I(m-2,3), R\big),\ \big(I(m-2,4), R\big),\ \big(2I(0,(m+1) \bmod 2+1), P(m-3,2)\big)\\
I(n_2,2):&\ \big(I(n_2-m+i+2,(m+i+1) \bmod 2+1), R\big),\ 1 \leq i \leq m-3\\
&\ \big(I(n_2,3), R\big),\ \big(I(n_2,4), R\big),\ \big((v+1)I, vP\big)
\end{aligned}$$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 1$, for $n \geq 0$. In the formulas below $0 \leq n_1 < m-2$ and $n_2 > m-2$:

$$\begin{aligned}
I(n_1,3):&\ \big(I(i,(n_1+i) \bmod 2+3), R\big),\ 0 \leq i \leq n_1-1\\
I(m-2,3):&\ \big(I(i,(m+i) \bmod 2+3), R\big),\ 1 \leq i \leq m-3\\
&\ \big(I(0,1), R\big),\ \big(I(0,2), R\big),\ \big(2I(0, m \bmod 2+3), P(m-3,3)\big)\\
I(n_2,3):&\ \big(I(n_2-m+i+2,(m+i) \bmod 2+3), R\big),\ 1 \leq i \leq m-3\\
&\ \big(I(n_2-m+2,1), R\big),\ \big(I(n_2-m+2,2), R\big),\ \big((v+1)I, vP\big)
\end{aligned}$$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 1$, for $n \geq 0$. In the formulas below $0 \leq n_1 < m-2$ and $n_2 > m-2$:

$$\begin{aligned}
I(n_1,4):&\ \big(I(i,(n_1+i+1) \bmod 2+3), R\big),\ 0 \leq i \leq n_1-1\\
I(m-2,4):&\ \big(I(i,(m+i+1) \bmod 2+3), R\big),\ 1 \leq i \leq m-3\\
&\ \big(I(0,1), R\big),\ \big(I(0,2), R\big),\ \big(2I(0,(m+1) \bmod 2+3), P(m-3,4)\big)\\
I(n_2,4):&\ \big(I(n_2-m+i+2,(m+i+1) \bmod 2+3), R\big),\ 1 \leq i \leq m-3\\
&\ \big(I(n_2-m+2,1), R\big),\ \big(I(n_2-m+2,2), R\big),\ \big((v+1)I, vP\big)
\end{aligned}$$

Modules of the form $I(n,j)$, for $5 \leq j \leq m+1$

Defect: $\partial I(n,j) = 2$, for $n \geq 0$. In the formulas below $0 \leq n_1 < j-4$ and $n_2 > j-4$:

$$\begin{aligned}
I(n_1,j):&\ \big(I(n_1,i), R\big),\ j+1 \leq i \leq m+1\\
&\ \big(I(i-j+n_1,i), R\big),\ j-n_1 \leq i \leq j-1\\
&\ \big(I(n_1,3), I(m-j+n_1+2,(m+j+1) \bmod 2+3)\big),\ \big(I(n_1,4), I(m-j+n_1+2,(m+j) \bmod 2+3)\big)\\
I(j-4,j):&\ \big(I(j-4,i), R\big),\ j+1 \leq i \leq m+1\\
&\ \big(I(i-4,i), R\big),\ 5 \leq i \leq j-1\\
&\ \big(I(j-4,3), I(m-2,(m+j+1) \bmod 2+3)\big),\ \big(I(j-4,4), I(m-2,(m+j) \bmod 2+3)\big),
\end{aligned}$$

$$\begin{aligned} & \big(I(0,1), I(j-4,(j+1) \bmod 2+1)\big),\ \big(I(0,2), I(j-4, j \bmod 2+1)\big) \\ I(n_2,j): & \big(I(n_2,i), R\big),\ j+1 \le i \le m+1 \\ & \big(I(n_2-j+i,i), R\big),\ 5 \le i \le j-1 \\ & \big(I(n_2,3), I(m-2,(m+j+1) \bmod 2+3)\big),\ \big(I(n_2,4), I(m-2,(m+j) \bmod 2+3)\big), \\ & \big(I(n_2-j+4,1), I(j-4,(j+1) \bmod 2+1)\big),\ \big(I(n_2-j+4,2), I(j-4, j \bmod 2+1)\big) \end{aligned}$$

Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_m)}$

$\underline{\dim} R_1^1(1) = \begin{smallmatrix}0\\0\end{smallmatrix}\, 0\ 0 \cdots 0\ 1\, \begin{smallmatrix}0\\0\end{smallmatrix}$, $\underline{\dim} R_1^2(1) = \begin{smallmatrix}1\\1\end{smallmatrix}\, 1\ 1\ 1 \cdots 1\, \begin{smallmatrix}1\\1\end{smallmatrix}$, $\underline{\dim} R_1^3(1) = \begin{smallmatrix}0\\0\end{smallmatrix}\, 1\ 0\ 0 \cdots 0\, \begin{smallmatrix}0\\0\end{smallmatrix}$,

$\underline{\dim} R_1^4(1) = \begin{smallmatrix}0\\0\end{smallmatrix}\, 0\ 1\ 0 \cdots 0\, \begin{smallmatrix}0\\0\end{smallmatrix}$, $\ldots$, $\underline{\dim} R_1^{m-2}(1) = \begin{smallmatrix}0\\0\end{smallmatrix}\, 0 \cdots 0\ 1\ 0\, \begin{smallmatrix}0\\0\end{smallmatrix}$

For all $1 \le t \le m-3$ and $1 \le l \le m-2$ we have the following Schofield pairs for non-homogeneous regulars of the form $R_1^l(t)$, where $t' = (l-3) \bmod (m-2) + 1$:

$$\begin{aligned} R_1^l(t): & \big(R_1^{(l-1+i) \bmod (m-2)+1}(t-i), R_1^l(i)\big),\ 1 \le i \le t-1 \\ & \big(I, P(t'-m+t+1, j)\big),\ 1 \le j \le m-t+1,\ \text{if } t'-m+t+1 \ge 0, \\ & \text{where } I \text{ is a preinjective exceptional with } \underline{\dim} I = \underline{\dim} R_1^l(t) - \underline{\dim} P(t'-m+t+1, j) \end{aligned}$$

The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_m)}$

$\underline{\dim} R_\infty^1(1) = \begin{smallmatrix}1\\0\end{smallmatrix}\, 1\ 1 \cdots 1\, \begin{smallmatrix}0\\1\end{smallmatrix}$, $\underline{\dim} R_\infty^2(1) = \begin{smallmatrix}0\\1\end{smallmatrix}\, 1\ 1 \cdots 1\, \begin{smallmatrix}1\\0\end{smallmatrix}$

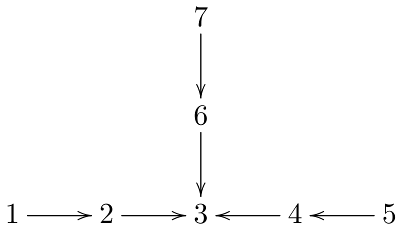
$$\begin{aligned} R_\infty^1(1): & \big(I(m-i-3,(m+i) \bmod 2+3), P(i, 2-(i \bmod 2))\big),\ 0 \le i \le m-3 \\ R_\infty^2(1): & \big(I(m-i-3,(m+i+1) \bmod 2+3), P(i, 2-((i+1) \bmod 2))\big),\ 0 \le i \le m-3 \end{aligned}$$

The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_m)}$

$\underline{\dim} R_0^1(1) = \begin{smallmatrix}1\\0\end{smallmatrix}\, 1\ 1 \cdots 1\, \begin{smallmatrix}1\\0\end{smallmatrix}$, $\underline{\dim} R_0^2(0) = \begin{smallmatrix}0\\1\end{smallmatrix}\, 1\ 1 \cdots 1\, \begin{smallmatrix}0\\1\end{smallmatrix}$

$$\begin{aligned} R_0^1(1): & \big(I(m-i-3,(m+i+1) \bmod 2+3), P(i, 2-(i \bmod 2))\big),\ 0 \le i \le m-3 \\ R_0^2(1): & \big(I(m-i-3,(m+i) \bmod 2+3), P(i, 2-((i+1) \bmod 2))\big),\ 0 \le i \le m-3 \end{aligned}$$

A.14 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{E}}_6)$ – $\delta = \begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$



$$C_{\Delta(\widetilde{\mathbb{E}}_6)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \Phi_{\Delta(\widetilde{\mathbb{E}}_6)} = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1) : \big(I(0,1), P(0,2)\big),\ \big(I(0,2), P(0,3)\big)$
$P(1,1) : \big(I(1,7), P(0,4)\big),\ \big(I(1,5), P(0,6)\big)$
$P(2,1) : \big(I(2,1), P(0,2)\big),\ \big(R_0^2(1), P(0,5)\big),\ \big(R_1^2(1), P(0,7)\big),\ \big(I(0,7), P(1,4)\big),\ \big(I(0,5), P(1,6)\big)$
$P(3,1) : \big(R_\infty^1(1), P(0,1)\big),\ \big(I(1,1), P(1,2)\big),\ \big(R_0^3(1), P(1,5)\big),\ \big(R_1^3(1), P(1,7)\big)$
$P(4,1) : \big(R_1^3(2), P(0,5)\big),\ \big(R_0^3(2), P(0,7)\big),\ \big(R_\infty^2(1), P(1,1)\big),\ \big(I(0,1), P(2,2)\big),\ \big(R_0^1(1), P(2,5)\big)$
$\big(R_1^1(1), P(2,7)\big)$
$P(5,1) : \big(R_1^1(2), P(1,5)\big),\ \big(R_0^1(2), P(1,7)\big),\ \big(R_\infty^1(1), P(2,1)\big),\ \big(R_0^2(1), P(3,5)\big),\ \big(R_1^2(1), P(3,7)\big)$
$P(6,1) : \big(R_1^2(2), P(2,5)\big),\ \big(R_0^2(2), P(2,7)\big),\ \big(R_\infty^2(1), P(3,1)\big),\ \big(R_0^3(1), P(4,5)\big),\ \big(R_1^3(1), P(4,7)\big)$
$\big(I(5,1), 2P(0,1)\big)$
$P(n,1) : \big(R_1^{(n-5) \bmod 3+1}(2), P(n-4,5)\big),\ \big(R_0^{(n-5) \bmod 3+1}(2), P(n-4,7)\big),\ \big(R_\infty^{(n-5) \bmod 2+1}(1), P(n-3,1)\big)$
$\big(R_0^{(n-4) \bmod 3+1}(1), P(n-2,5)\big),\ \big(R_1^{(n-4) \bmod 3+1}(1), P(n-2,7)\big),\ \big(uI, (u+1)P\big),\ n > 6$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -2$, for $n \geq 0$.

$P(0,2) : \big(I(1,1), P(0,3)\big)$
$P(1,2) : \big(P(1,1), P(0,1)\big),\ \big(R_1^3(1), P(0,4)\big),\ \big(R_0^3(1), P(0,6)\big),\ \big(I(0,1), P(1,3)\big)$
$P(2,2) : \big(P(3,7), P(0,5)\big),\ \big(P(3,5), P(0,7)\big),\ \big(P(2,1), P(1,1)\big),\ \big(R_1^1(1), P(1,4)\big),\ \big(R_0^1(1), P(1,6)\big)$
$P(3,2) : \big(P(5,1), P(0,1)\big),\ \big(P(4,7), P(1,5)\big),\ \big(P(4,5), P(1,7)\big),\ \big(P(3,1), P(2,1)\big),\ \big(R_1^2(1), P(2,4)\big)$
$\big(R_0^2(1), P(2,6)\big)$
$P(n,2) : \big(P(n+2,1), P(n-3,1)\big),\ \big(P(n+1,7), P(n-2,5)\big),\ \big(P(n+1,5), P(n-2,7)\big)$
$\big(P(n,1), P(n-1,1)\big),\ \big(R_1^{(n-2) \bmod 3+1}(1), P(n-1,4)\big),\ \big(R_0^{(n-2) \bmod 3+1}(1), P(n-1,6)\big),\ n > 3$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3) = -3$, for $n \geq 0$.

$P(0,3)$: –
$P(1,3) : \big(P(1,1), P(0,2)\big),\ \big(P(1,5), P(0,4)\big),\ \big(P(1,7), P(0,6)\big)$
$P(2,3) : \big(P(2,2), P(0,1)\big),\ \big(P(2,4), P(0,5)\big),\ \big(P(2,6), P(0,7)\big),\ \big(P(2,1), P(1,2)\big),\ \big(P(2,5), P(1,4)\big)$
$\big(P(2,7), P(1,6)\big)$
$P(n,3) : \big(P(n,2), P(n-2,1)\big),\ \big(P(n,4), P(n-2,5)\big),\ \big(P(n,6), P(n-2,7)\big)$
$\big(P(n,1), P(n-1,2)\big),\ \big(P(n,5), P(n-1,4)\big),\ \big(P(n,7), P(n-1,6)\big),\ n > 2$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4) = -2$, for $n \geq 0$.

$P(0,4) : \big(I(1,5), P(0,3)\big)$
$P(1,4) : \big(R_0^1(1), P(0,2)\big),\ \big(P(1,5), P(0,5)\big),\ \big(R_1^2(1), P(0,6)\big),\ \big(I(0,5), P(1,3)\big)$
$P(2,4) : \big(P(3,7), P(0,1)\big),\ \big(P(3,1), P(0,7)\big),\ \big(R_0^2(1), P(1,2)\big),\ \big(P(2,5), P(1,5)\big),\ \big(R_1^3(1), P(1,6)\big)$
$P(3,4) : \big(P(5,5), P(0,5)\big),\ \big(P(4,7), P(1,1)\big),\ \big(P(4,1), P(1,7)\big),\ \big(R_0^3(1), P(2,2)\big),\ \big(P(3,5), P(2,5)\big)$
$\big(R_1^1(1), P(2,6)\big)$
$P(n,4) : \big(P(n+2,5), P(n-3,5)\big),\ \big(P(n+1,7), P(n-2,1)\big),\ \big(P(n+1,1), P(n-2,7)\big)$
$\big(R_0^{(n-1) \bmod 3+1}(1), P(n-1,2)\big),\ \big(P(n,5), P(n-1,5)\big),\ \big(R_1^{(n-3) \bmod 3+1}(1), P(n-1,6)\big),\ n > 3$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -1$, for $n \geq 0$.

$P(0,5) : \big(I(0,4), P(0,3)\big),\ \big(I(0,5), P(0,4)\big)$
$P(1,5) : \big(I(1,7), P(0,2)\big),\ \big(I(1,1), P(0,6)\big)$
$P(2,5) : \big(R_1^1(1), P(0,1)\big),\ \big(I(2,5), P(0,4)\big),\ \big(R_0^3(1), P(0,7)\big),\ \big(I(0,7), P(1,2)\big),\ \big(I(0,1), P(1,6)\big)$
$P(3,5) : \big(R_\infty^1(1), P(0,5)\big),\ \big(R_1^2(1), P(1,1)\big),\ \big(I(1,5), P(1,4)\big),\ \big(R_0^1(1), P(1,7)\big)$
$P(4,5) : \big(R_0^1(2), P(0,1)\big),\ \big(R_1^2(2), P(0,7)\big),\ \big(R_\infty^2(1), P(1,5)\big),\ \big(R_1^3(1), P(2,1)\big),\ \big(I(0,5), P(2,4)\big)$
$\big(R_0^2(1), P(2,7)\big)$
$P(5,5) : \big(R_0^2(2), P(1,1)\big),\ \big(R_1^3(2), P(1,7)\big),\ \big(R_\infty^1(1), P(2,5)\big),\ \big(R_1^1(1), P(3,1)\big),\ \big(R_0^3(1), P(3,7)\big)$
$P(6,5) : \big(R_0^3(2), P(2,1)\big),\ \big(R_1^1(2), P(2,7)\big),\ \big(R_\infty^2(1), P(3,5)\big),\ \big(R_1^2(1), P(4,1)\big),\ \big(R_0^1(1), P(4,7)\big)$
$\big(I(5,5), 2P(0,5)\big)$

$P(n,5): \big(R_0^{(n-4) \bmod 3+1}(2), P(n-4,1)\big),\ \big(R_1^{(n-6) \bmod 3+1}(2), P(n-4,7)\big),\ \big(R_\infty^{(n-5) \bmod 2+1}(1), P(n-3,5)\big)$
$\big(R_1^{(n-5) \bmod 3+1}(1), P(n-2,1)\big),\ \big(R_0^{(n-6) \bmod 3+1}(1), P(n-2,7)\big),\ \big(uI, (u+1)P\big),\ n > 6$

Modules of the form $P(n,6)$

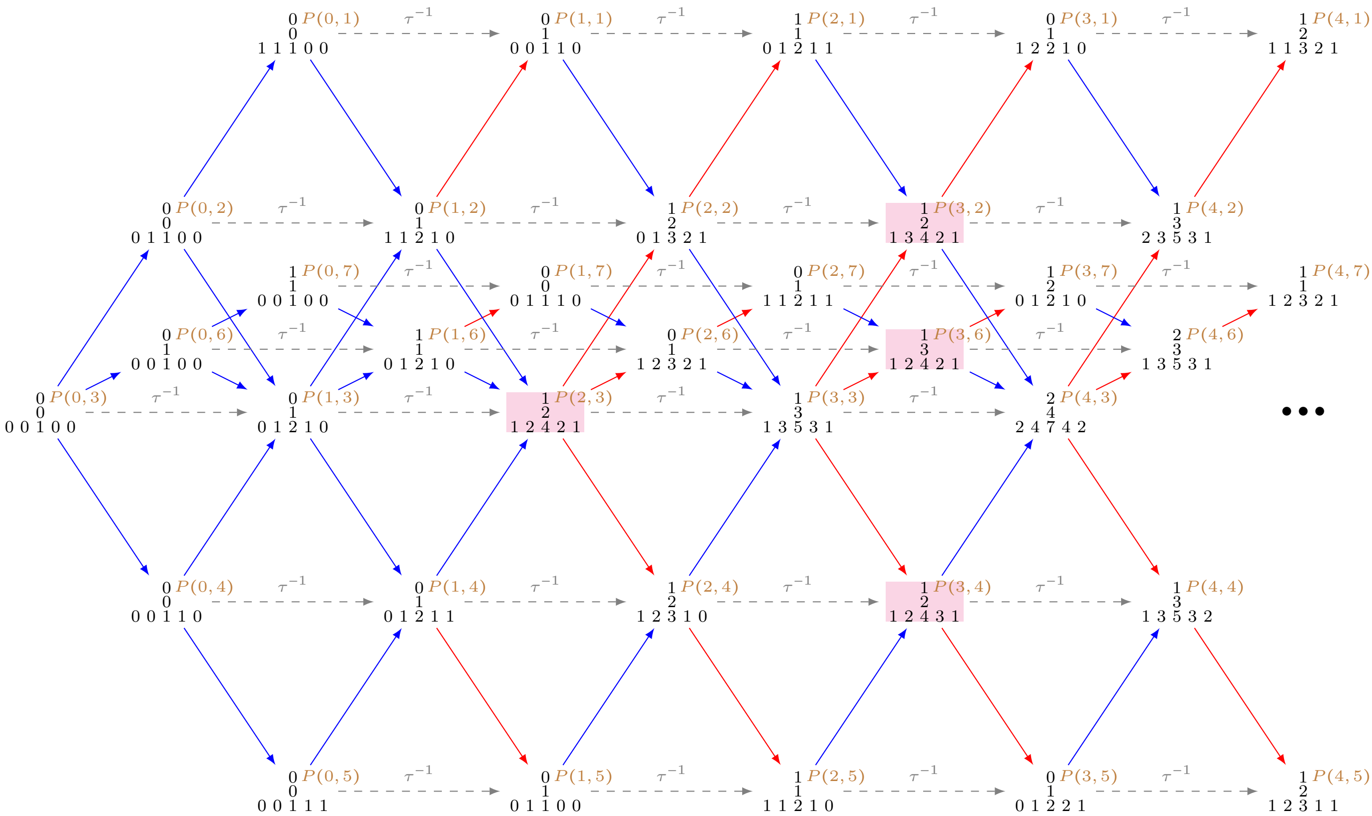
Defect: $\partial P(n,6) = -2$, for $n \geq 0$.

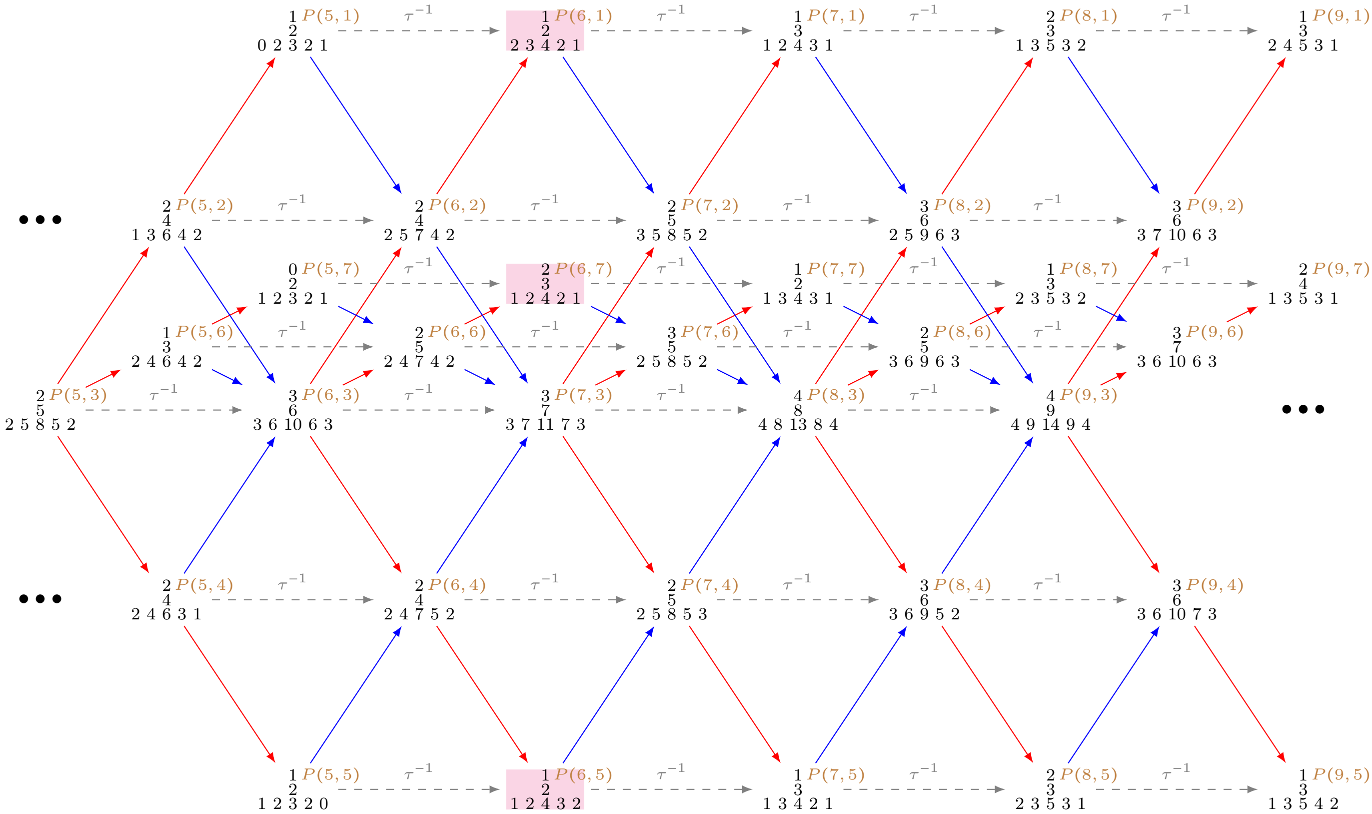
$P(0,6): \big(I(1,7), P(0,3)\big)$
$P(1,6): \big(R_1^1(1), P(0,2)\big),\ \big(R_0^2(1), P(0,4)\big),\ \big(P(1,7), P(0,7)\big),\ \big(I(0,7), P(1,3)\big)$
$P(2,6): \big(P(3,5), P(0,1)\big),\ \big(P(3,1), P(0,5)\big),\ \big(R_1^2(1), P(1,2)\big),\ \big(R_0^3(1), P(1,4)\big),\ \big(P(2,7), P(1,7)\big)$
$P(3,6): \big(P(5,7), P(0,7)\big),\ \big(P(4,5), P(1,1)\big),\ \big(P(4,1), P(1,5)\big),\ \big(R_1^3(1), P(2,2)\big),\ \big(R_0^1(1), P(2,4)\big)$
$\big(P(3,7), P(2,7)\big)$
$P(n,6): \big(P(n+2,7), P(n-3,7)\big),\ \big(P(n+1,5), P(n-2,1)\big),\ \big(P(n+1,1), P(n-2,5)\big)$
$\big(R_1^{(n-1) \bmod 3+1}(1), P(n-1,2)\big),\ \big(R_0^{(n-3) \bmod 3+1}(1), P(n-1,4)\big),\ \big(P(n,7), P(n-1,7)\big),\ n > 3$

Modules of the form $P(n,7)$

Defect: $\partial P(n,7) = -1$, for $n \geq 0$.

$P(0,7): \big(I(0,6), P(0,3)\big),\ \big(I(0,7), P(0,6)\big)$
$P(1,7): \big(I(1,5), P(0,2)\big),\ \big(I(1,1), P(0,4)\big)$
$P(2,7): \big(R_0^1(1), P(0,1)\big),\ \big(R_1^3(1), P(0,5)\big),\ \big(I(2,7), P(0,6)\big),\ \big(I(0,5), P(1,2)\big),\ \big(I(0,1), P(1,4)\big)$
$P(3,7): \big(R_\infty^1(1), P(0,7)\big),\ \big(R_0^2(1), P(1,1)\big),\ \big(R_1^1(1), P(1,5)\big),\ \big(I(1,7), P(1,6)\big)$
$P(4,7): \big(R_1^1(2), P(0,1)\big),\ \big(R_0^2(2), P(0,5)\big),\ \big(R_\infty^2(1), P(1,7)\big),\ \big(R_0^3(1), P(2,1)\big),\ \big(R_1^2(1), P(2,5)\big)$
$\big(I(0,7), P(2,6)\big)$
$P(5,7): \big(R_1^2(2), P(1,1)\big),\ \big(R_0^3(2), P(1,5)\big),\ \big(R_\infty^1(1), P(2,7)\big),\ \big(R_0^1(1), P(3,1)\big),\ \big(R_1^3(1), P(3,5)\big)$
$P(6,7): \big(R_1^3(2), P(2,1)\big),\ \big(R_0^1(2), P(2,5)\big),\ \big(R_\infty^2(1), P(3,7)\big),\ \big(R_0^2(1), P(4,1)\big),\ \big(R_1^1(1), P(4,5)\big)$
$\big(I(5,7), 2P(0,7)\big)$
$P(n,7): \big(R_1^{(n-4) \bmod 3+1}(2), P(n-4,1)\big),\ \big(R_0^{(n-6) \bmod 3+1}(2), P(n-4,5)\big),\ \big(R_\infty^{(n-5) \bmod 2+1}(1), P(n-3,7)\big)$
$\big(R_0^{(n-5) \bmod 3+1}(1), P(n-2,1)\big),\ \big(R_1^{(n-6) \bmod 3+1}(1), P(n-2,5)\big),\ \big(uI, (u+1)P\big),\ n > 6$


0 P(0,1) 0 1 1 1 0 0
0 P(1,1) 1 0 0 1 1 0
1 P(2,1) 1 0 1 2 1 1
0 P(3,1) 1 1 2 2 1 0
1 P(4,1) 2 1 1 3 2 1
0 P(0,2) 0 0 1 1 0 0
0 P(1,2) 1 1 1 2 1 0
1 P(2,2) 2 0 1 3 2 1
1 P(3,2) 2 1 3 4 2 1
1 P(4,2) 3 2 3 5 3 1
1 P(0,7) 1 0 0 1 0 0
0 P(1,7) 0 0 1 1 1 0
0 P(2,7) 1 1 1 2 1 1
1 P(3,7) 2 0 1 2 1 0
1 P(4,7) 1 1 2 3 2 1
0 P(0,6) 1 0 0 1 0 0
1 P(1,6) 1 0 1 2 1 0
0 P(2,6) 1 1 2 3 2 1
1 P(3,6) 3 1 2 4 2 1
2 P(4,6) 3 1 3 5 3 1
0 P(0,3) 0 0 0 1 0 0
0 P(1,3) 1 0 1 2 1 0
1 P(2,3) 2 1 2 4 2 1
1 P(3,3) 3 1 3 5 3 1
2 P(4,3) 4 2 4 7 4 2
0 P(0,4) 0 0 0 1 1 0
0 P(1,4) 1 0 1 2 1 1
1 P(2,4) 2 1 2 3 1 0
1 P(3,4) 2 1 2 4 3 1
1 P(4,4) 3 1 3 5 3 2
0 P(0,5) 0 0 0 1 1 1
0 P(1,5) 1 0 1 1 0 0
1 P(2,5) 1 1 1 2 1 0
0 P(3,5) 1 0 1 2 2 1
1 P(4,5) 2 1 2 3 1 1
τ^{-1}


1 P(5,1) 2 0 2 3 2 1
1 P(6,1) 2 2 3 4 2 1
1 P(7,1) 3 1 2 4 3 1
2 P(8,1) 3 1 3 5 3 2
1 P(9,1) 3 2 4 5 3 1
2 P(5,2) 4 1 3 6 4 2
2 P(6,2) 4 2 5 7 4 2
2 P(7,2) 5 3 5 8 5 2
3 P(8,2) 6 2 5 9 6 3
3 P(9,2) 6 3 7 10 6 3
0 P(5,7) 2 1 2 3 2 1
2 P(6,7) 3 1 2 4 2 1
1 P(7,7) 2 1 3 4 3 1
1 P(8,7) 3 2 3 5 3 2
2 P(9,7) 4 1 3 5 3 1
1 P(5,6) 3 2 4 6 4 2
2 P(6,6) 5 2 4 7 4 2
3 P(7,6) 5 2 5 8 5 2
2 P(8,6) 5 3 6 9 6 3
3 P(9,6) 7 3 6 10 6 3
2 P(5,3) 5 2 5 8 5 2
3 P(6,3) 6 3 6 10 6 3
3 P(7,3) 7 3 7 11 7 3
4 P(8,3) 8 4 8 13 8 4
4 P(9,3) 9 4 9 14 9 4
2 P(5,4) 4 2 4 6 3 1
2 P(6,4) 4 2 4 7 5 2
2 P(7,4) 5 2 5 8 5 3
3 P(8,4) 6 3 6 9 5 2
3 P(9,4) 6 3 6 10 7 3
1 P(5,5) 2 1 2 3 2 0
1 P(6,5) 2 1 2 4 3 2
1 P(7,5) 3 1 3 4 2 1
2 P(8,5) 3 2 3 5 3 1
1 P(9,5) 3 1 3 5 4 2
τ^{-1}

Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$I(0,1)$: –

$I(1,1)$: –

$I(2,1) : \big(I(0,4), P(0,7)\big),\ \big(I(0,5), R_1^1(1)\big),\ \big(I(0,6), P(0,5)\big),\ \big(I(0,7), R_0^1(1)\big)$

$I(3,1) : \big(I(0,1), R_\infty^1(1)\big),\ \big(I(0,2), P(1,1)\big),\ \big(I(1,5), R_1^3(1)\big),\ \big(I(1,7), R_0^3(1)\big)$

$I(4,1) : \big(I(0,5), R_0^2(2)\big),\ \big(I(0,7), R_1^2(2)\big),\ \big(I(1,1), R_\infty^2(1)\big),\ \big(I(1,2), P(0,1)\big),\ \big(I(2,5), R_1^2(1)\big)$
$\big(I(2,7), R_0^2(1)\big)$

$I(5,1) : \big(I(1,5), R_0^1(2)\big),\ \big(I(1,7), R_1^1(2)\big),\ \big(I(2,1), R_\infty^1(1)\big),\ \big(I(3,5), R_1^1(1)\big),\ \big(I(3,7), R_0^1(1)\big)$

$I(6,1) : \big(I(2,5), R_0^3(2)\big),\ \big(I(2,7), R_1^3(2)\big),\ \big(I(3,1), R_\infty^2(1)\big),\ \big(I(4,5), R_1^3(1)\big),\ \big(I(4,7), R_0^3(1)\big)$
$\big(2I(0,1), P(5,1)\big)$

$I(n,1) : \big(I(n-4,5), R_0^{(-n+8) \bmod 3+1}(2)\big),\ \big(I(n-4,7), R_1^{(-n+8) \bmod 3+1}(2)\big),\ \big(I(n-3,1), R_\infty^{(-n+7) \bmod 2+1}(1)\big)$
$\big(I(n-2,5), R_1^{(-n+8) \bmod 3+1}(1)\big),\ \big(I(n-2,7), R_0^{(-n+8) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 6$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 2$, for $n \geq 0$.

$I(0,2) : \big(I(0,1), I(1,1)\big)$

$I(1,2) : \big(I(0,4), R_0^2(1)\big),\ \big(I(0,5), I(3,7)\big),\ \big(I(0,6), R_1^2(1)\big),\ \big(I(0,7), I(3,5)\big),\ \big(I(1,1), I(2,1)\big)$

$I(2,2) : \big(I(0,1), I(5,1)\big),\ \big(I(1,4), R_0^1(1)\big),\ \big(I(1,5), I(4,7)\big),\ \big(I(1,6), R_1^1(1)\big),\ \big(I(1,7), I(4,5)\big)$
$\big(I(2,1), I(3,1)\big)$

$I(n,2) : \big(I(n-2,1), I(n+3,1)\big),\ \big(I(n-1,4), R_0^{(-n+2) \bmod 3+1}(1)\big),\ \big(I(n-1,5), I(n+2,7)\big)$
$\big(I(n-1,6), R_1^{(-n+2) \bmod 3+1}(1)\big),\ \big(I(n-1,7), I(n+2,5)\big),\ \big(I(n,1), I(n+1,1)\big),\ n > 2$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 3$, for $n \geq 0$.

$I(0,3) : \big(I(0,1), I(1,2)\big),\ \big(I(0,2), I(2,1)\big),\ \big(I(0,4), I(2,5)\big),\ \big(I(0,5), I(1,4)\big),\ \big(I(0,6), I(2,7)\big)$
$\big(I(0,7), I(1,6)\big)$

$I(n,3) : \big(I(n,1), I(n+1,2)\big),\ \big(I(n,2), I(n+2,1)\big),\ \big(I(n,4), I(n+2,5)\big)$
$\big(I(n,5), I(n+1,4)\big),\ \big(I(n,6), I(n+2,7)\big),\ \big(I(n,7), I(n+1,6)\big),\ n > 0$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 2$, for $n \geq 0$.

$$I(0,4) : \big(I(0,5), I(1,5)\big)$$
$$I(1,4) : \big(I(0,1), I(3,7)\big),\ \big(I(0,2), R_1^1(1)\big),\ \big(I(0,6), R_0^3(1)\big),\ \big(I(0,7), I(3,1)\big),\ \big(I(1,5), I(2,5)\big)$$
$$I(2,4) : \big(I(0,5), I(5,5)\big),\ \big(I(1,1), I(4,7)\big),\ \big(I(1,2), R_1^3(1)\big),\ \big(I(1,6), R_0^2(1)\big),\ \big(I(1,7), I(4,1)\big)$$
$$\big(I(2,5), I(3,5)\big)$$
$$I(n,4) : \big(I(n-2,5), I(n+3,5)\big),\ \big(I(n-1,1), I(n+2,7)\big),\ \big(I(n-1,2), R_1^{(-n+4) \bmod 3+1}(1)\big)$$
$$\big(I(n-1,6), R_0^{(-n+3) \bmod 3+1}(1)\big),\ \big(I(n-1,7), I(n+2,1)\big),\ \big(I(n,5), I(n+1,5)\big),\ n > 2$$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 1$, for $n \geq 0$.

$$I(0,5) : -$$
$$I(1,5) : -$$
$$I(2,5) : \big(I(0,1), R_0^2(1)\big),\ \big(I(0,2), P(0,7)\big),\ \big(I(0,6), P(0,1)\big),\ \big(I(0,7), R_1^3(1)\big)$$
$$I(3,5) : \big(I(0,4), P(1,5)\big),\ \big(I(0,5), R_\infty^1(1)\big),\ \big(I(1,1), R_0^1(1)\big),\ \big(I(1,7), R_1^2(1)\big)$$
$$I(4,5) : \big(I(0,1), R_1^1(2)\big),\ \big(I(0,7), R_0^3(2)\big),\ \big(I(1,4), P(0,5)\big),\ \big(I(1,5), R_\infty^2(1)\big),\ \big(I(2,1), R_0^3(1)\big)$$
$$\big(I(2,7), R_1^1(1)\big)$$
$$I(5,5) : \big(I(1,1), R_1^3(2)\big),\ \big(I(1,7), R_0^2(2)\big),\ \big(I(2,5), R_\infty^1(1)\big),\ \big(I(3,1), R_0^2(1)\big),\ \big(I(3,7), R_1^3(1)\big)$$
$$I(6,5) : \big(I(2,1), R_1^2(2)\big),\ \big(I(2,7), R_0^1(2)\big),\ \big(I(3,5), R_\infty^2(1)\big),\ \big(I(4,1), R_0^1(1)\big),\ \big(I(4,7), R_1^2(1)\big)$$
$$\big(2I(0,5), P(5,5)\big)$$
$$I(n,5) : \big(I(n-4,1), R_1^{(-n+7) \bmod 3+1}(2)\big),\ \big(I(n-4,7), R_0^{(-n+6) \bmod 3+1}(2)\big),\ \big(I(n-3,5), R_\infty^{(-n+7) \bmod 2+1}(1)\big)$$
$$\big(I(n-2,1), R_0^{(-n+6) \bmod 3+1}(1)\big),\ \big(I(n-2,7), R_1^{(-n+7) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 6$$

Modules of the form $I(n,6)$

Defect: $\partial I(n,6) = 2$, for $n \geq 0$.

$$I(0,6) : \big(I(0,7), I(1,7)\big)$$
$$I(1,6) : \big(I(0,1), I(3,5)\big),\ \big(I(0,2), R_0^1(1)\big),\ \big(I(0,4), R_1^3(1)\big),\ \big(I(0,5), I(3,1)\big),\ \big(I(1,7), I(2,7)\big)$$
$$I(2,6) : \big(I(0,7), I(5,7)\big),\ \big(I(1,1), I(4,5)\big),\ \big(I(1,2), R_0^3(1)\big),\ \big(I(1,4), R_1^2(1)\big),\ \big(I(1,5), I(4,1)\big)$$
$$\big(I(2,7), I(3,7)\big)$$
$$I(n,6) : \big(I(n-2,7), I(n+3,7)\big),\ \big(I(n-1,1), I(n+2,5)\big),\ \big(I(n-1,2), R_0^{(-n+4) \bmod 3+1}(1)\big)$$
$$\big(I(n-1,4), R_1^{(-n+3) \bmod 3+1}(1)\big),\ \big(I(n-1,5), I(n+2,1)\big),\ \big(I(n,7), I(n+1,7)\big),\ n > 2$$

Modules of the form $I(n,7)$

Defect: $\partial I(n,7) = 1$, for $n \geq 0$.

$I(0,7)$: –

$I(1,7)$: –

$I(2,7) : \big(I(0,1), R_1^2(1)\big),\ \big(I(0,2), P(0,5)\big),\ \big(I(0,4), P(0,1)\big),\ \big(I(0,5), R_0^3(1)\big)$

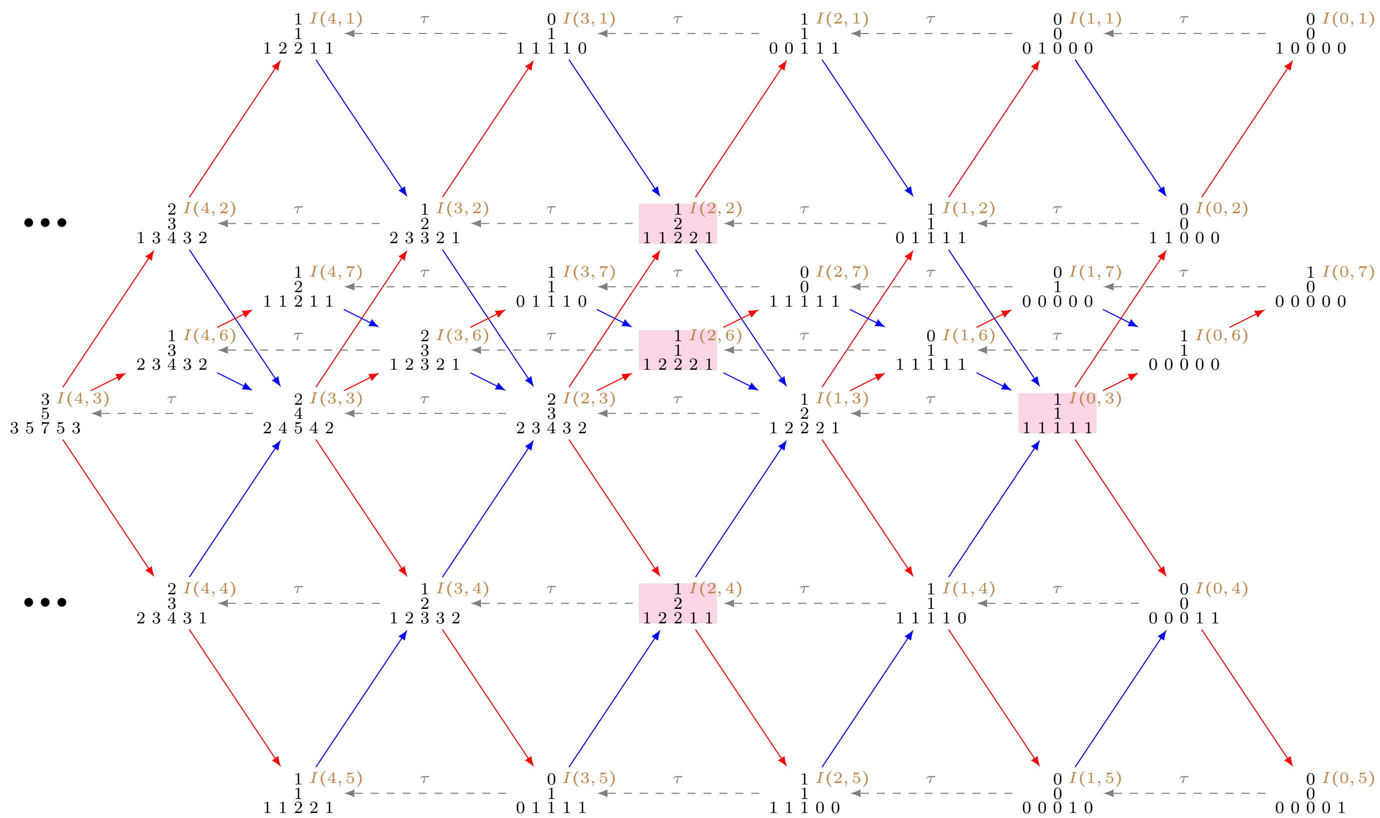
$I(3,7) : \big(I(0,6), P(1,7)\big),\ \big(I(0,7), R_\infty^1(1)\big),\ \big(I(1,1), R_1^1(1)\big),\ \big(I(1,5), R_0^2(1)\big)$

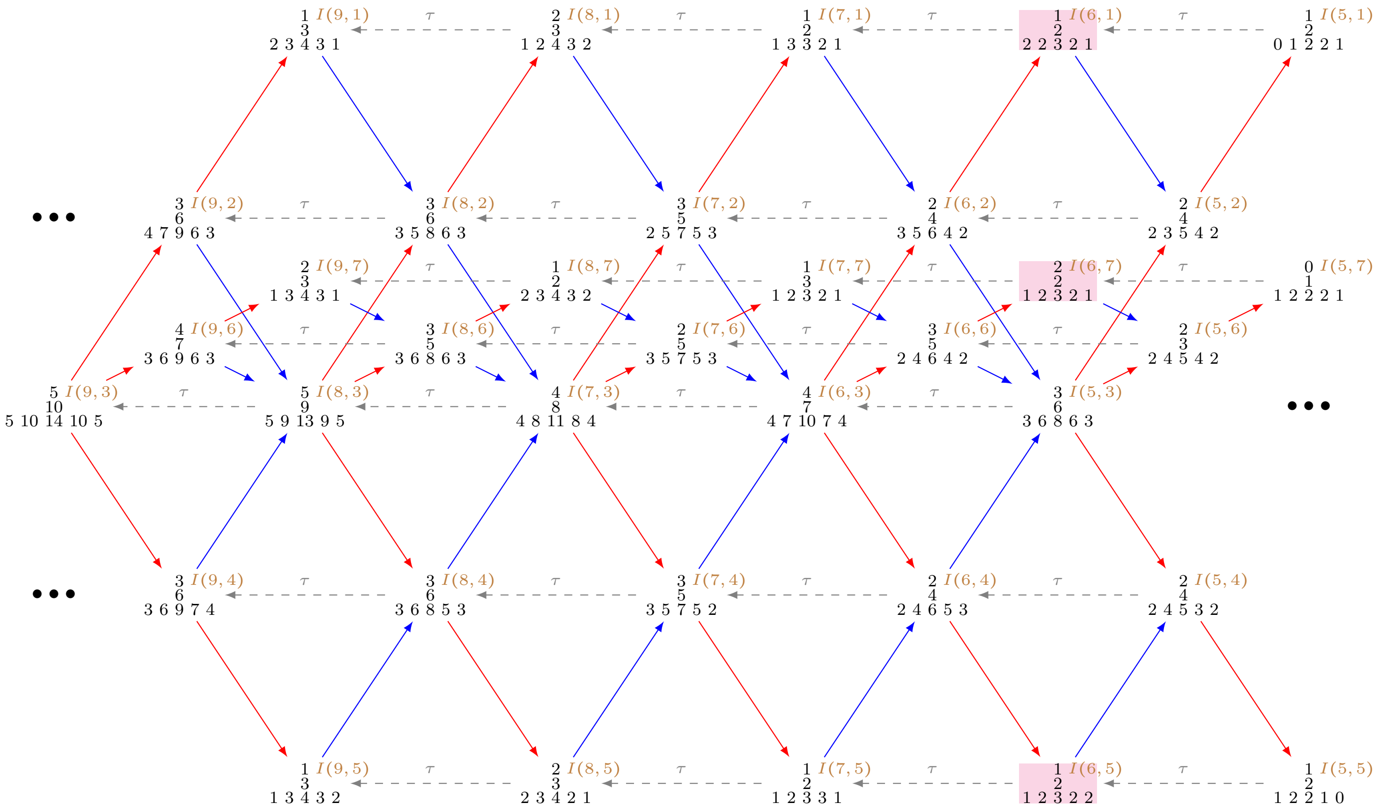
$I(4,7) : \big(I(0,1), R_0^1(2)\big),\ \big(I(0,5), R_1^3(2)\big),\ \big(I(1,6), P(0,7)\big),\ \big(I(1,7), R_\infty^2(1)\big),\ \big(I(2,1), R_1^3(1)\big)$
$\big(I(2,5), R_0^1(1)\big)$

$I(5,7) : \big(I(1,1), R_0^3(2)\big),\ \big(I(1,5), R_1^2(2)\big),\ \big(I(2,7), R_\infty^1(1)\big),\ \big(I(3,1), R_1^2(1)\big),\ \big(I(3,5), R_0^3(1)\big)$

$I(6,7) : \big(I(2,1), R_0^2(2)\big),\ \big(I(2,5), R_1^1(2)\big),\ \big(I(3,7), R_\infty^2(1)\big),\ \big(I(4,1), R_1^1(1)\big),\ \big(I(4,5), R_0^2(1)\big)$
$\big(2I(0,7), P(5,7)\big)$

$I(n,7) : \big(I(n-4,1), R_0^{(-n+7) \bmod 3+1}(2)\big),\ \big(I(n-4,5), R_1^{(-n+6) \bmod 3+1}(2)\big),\ \big(I(n-3,7), R_\infty^{(-n+7) \bmod 2+1}(1)\big)$
$\big(I(n-2,1), R_1^{(-n+6) \bmod 3+1}(1)\big),\ \big(I(n-2,5), R_0^{(-n+7) \bmod 3+1}(1)\big),\ \big((v+1)I, vP\big),\ n > 6$





Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{E}}_6)}$

$R_1^1(1) : \big(I(0,6), P(0,4)\big),\ \big(I(1,5), P(0,7)\big),\ \big(I(0,7), P(1,1)\big)$

$R_1^1(2) : \big(R_1^2(1), R_1^1(1)\big),\ \big(I(3,7), P(0,5)\big),\ \big(I(2,1), P(1,7)\big),\ \big(I(1,5), P(2,1)\big),\ \big(I(0,7), P(3,5)\big)$

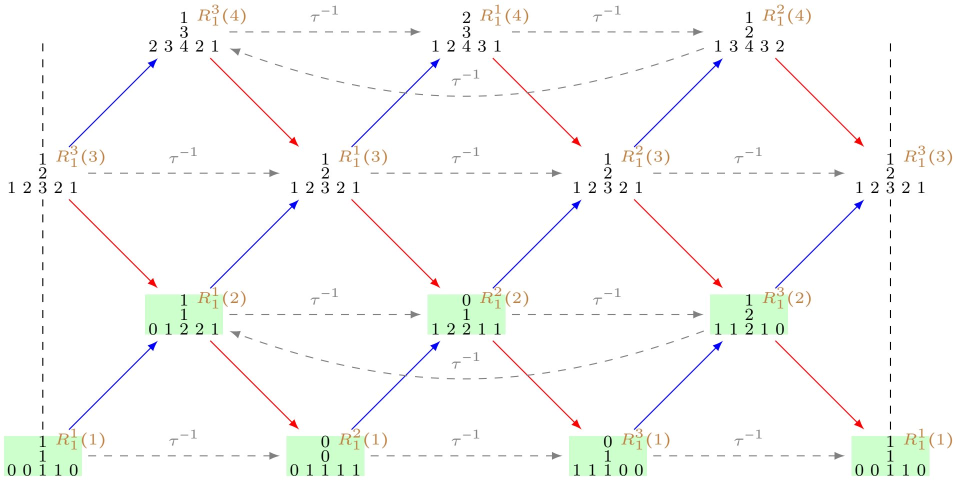
$R_1^2(1) : \big(I(0,4), P(0,2)\big),\ \big(I(1,1), P(0,5)\big),\ \big(I(0,5), P(1,7)\big)$

$R_1^2(2) : \big(R_1^3(1), R_1^2(1)\big),\ \big(I(3,5), P(0,1)\big),\ \big(I(2,7), P(1,5)\big),\ \big(I(1,1), P(2,7)\big),\ \big(I(0,5), P(3,1)\big)$

$R_1^3(1) : \big(I(1,7), P(0,1)\big),\ \big(I(0,2), P(0,6)\big),\ \big(I(0,1), P(1,5)\big)$

$R_1^3(2) : \big(R_1^1(1), R_1^3(1)\big),\ \big(I(3,1), P(0,7)\big),\ \big(I(2,5), P(1,1)\big),\ \big(I(1,7), P(2,5)\big),\ \big(I(0,1), P(3,7)\big)$

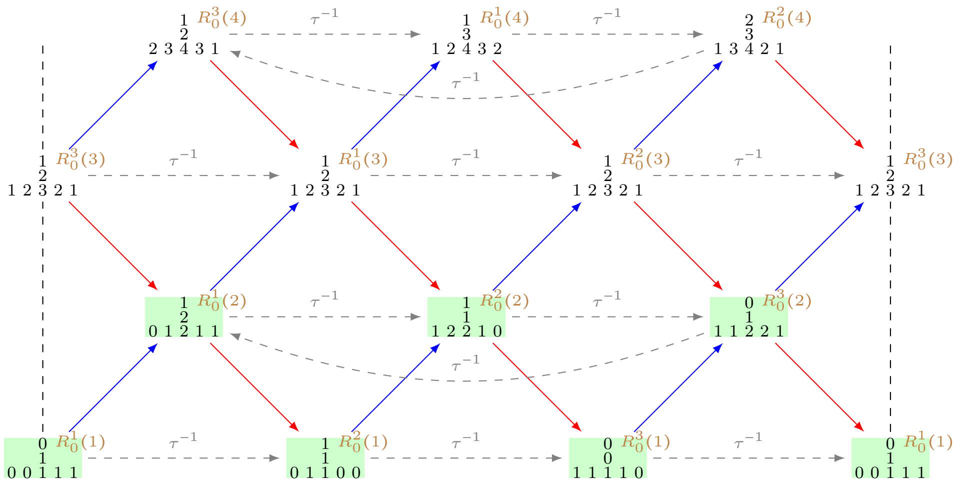
$\mathcal{T}_1^{\Delta(\widetilde{\mathbb{E}}_6)}$



The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{E}}_6)}$

$R_0^1(1) : \big(I(1,7), P(0,5)\big),\ \big(I(0,4), P(0,6)\big),\ \big(I(0,5), P(1,1)\big)$
$R_0^1(2) : \big(R_0^2(1), R_0^1(1)\big),\ \big(I(3,5), P(0,7)\big),\ \big(I(2,1), P(1,5)\big),\ \big(I(1,7), P(2,1)\big),\ \big(I(0,5), P(3,7)\big)$
$R_0^2(1) : \big(I(0,6), P(0,2)\big),\ \big(I(1,1), P(0,7)\big),\ \big(I(0,7), P(1,5)\big)$
$R_0^2(2) : \big(R_0^3(1), R_0^2(1)\big),\ \big(I(3,7), P(0,1)\big),\ \big(I(2,5), P(1,7)\big),\ \big(I(1,1), P(2,5)\big),\ \big(I(0,7), P(3,1)\big)$
$R_0^3(1) : \big(I(1,5), P(0,1)\big),\ \big(I(0,2), P(0,4)\big),\ \big(I(0,1), P(1,7)\big)$
$R_0^3(2) : \big(R_0^1(1), R_0^3(1)\big),\ \big(I(3,1), P(0,5)\big),\ \big(I(2,7), P(1,1)\big),\ \big(I(1,5), P(2,7)\big),\ \big(I(0,1), P(3,5)\big)$

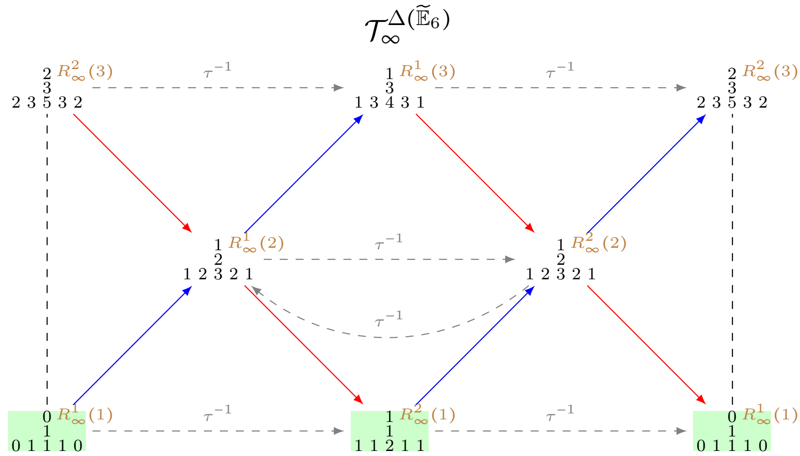
$\mathcal{T}_0^{\Delta(\widetilde{\mathbb{E}}_6)}$



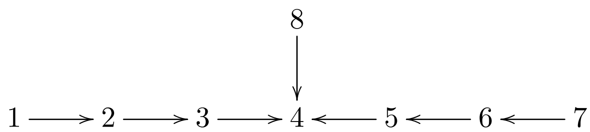
The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{E}}_6)}$

$R_\infty^1(1) : \big(I(1,1), P(1,1)\big),\ \big(I(1,5), P(1,5)\big),\ \big(I(1,7), P(1,7)\big)$

$R_\infty^2(1) : \big(I(2,1), P(0,1)\big),\ \big(I(2,5), P(0,5)\big),\ \big(I(2,7), P(0,7)\big),\ \big(I(0,1), P(2,1)\big),\ \big(I(0,5), P(2,5)\big)$
$\big(I(0,7), P(2,7)\big)$



A.15 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{E}}_7)$ – $\delta = {}_{1\,2\,3\,\overset{2}{4}\,3\,2\,1}$



$$C_{\Delta(\widetilde{\mathbb{E}}_7)} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&0\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \qquad \Phi_{\Delta(\widetilde{\mathbb{E}}_7)} = \begin{bmatrix} 0&0&0&-1&1&0&0&1\\ 1&0&0&-1&1&0&0&1\\ 0&1&0&-1&1&0&0&1\\ 0&0&1&-1&1&0&0&1\\ 0&0&1&-1&0&1&0&1\\ 0&0&1&-1&0&0&1&1\\ 0&0&1&-1&0&0&0&1\\ 0&0&1&-1&1&0&0&0 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -1$, for $n \geq 0$.

$P(0,1): \big(I(0,1), P(0,2)\big),\ \big(I(0,2), P(0,3)\big),\ \big(I(0,3), P(0,4)\big)$
$P(1,1): \big(I(0,8), P(0,5)\big),\ \big(I(2,7), P(0,8)\big)$
$P(2,1): \big(I(1,6), P(0,3)\big),\ \big(I(2,1), P(0,6)\big),\ \big(I(1,7), P(1,8)\big)$
$P(3,1): \big(I(3,1), P(0,2)\big),\ \big(R_1^3(1), P(0,7)\big),\ \big(I(4,7), P(0,8)\big),\ \big(I(0,6), P(1,3)\big),\ \big(I(1,1), P(1,6)\big)$
$\big(I(0,7), P(2,8)\big)$
$P(4,1): \big(R_0^1(1), P(0,1)\big),\ \big(I(2,1), P(1,2)\big),\ \big(R_1^4(1), P(1,7)\big),\ \big(I(3,7), P(1,8)\big),\ \big(I(0,1), P(2,6)\big)$
$P(5,1): \big(I(5,1), P(0,6)\big),\ \big(R_0^2(1), P(1,1)\big),\ \big(I(1,1), P(2,2)\big),\ \big(R_1^1(1), P(2,7)\big),\ \big(I(2,7), P(2,8)\big)$
$P(6,1): \big(R_1^1(2), P(0,1)\big),\ \big(R_\infty^2(1), P(0,7)\big),\ \big(I(4,1), P(1,6)\big),\ \big(R_0^3(1), P(2,1)\big),\ \big(I(0,1), P(3,2)\big)$
$\big(R_1^2(1), P(3,7)\big),\ \big(I(1,7), P(3,8)\big)$
$P(7,1): \big(R_1^2(2), P(1,1)\big),\ \big(R_\infty^1(1), P(1,7)\big),\ \big(I(3,1), P(2,6)\big),\ \big(R_0^1(1), P(3,1)\big),\ \big(R_1^3(1), P(4,7)\big)$
$\big(I(0,7), P(4,8)\big)$
$P(8,1): \big(R_0^1(2), P(0,1)\big),\ \big(R_1^3(2), P(2,1)\big),\ \big(R_\infty^2(1), P(2,7)\big),\ \big(I(2,1), P(3,6)\big),\ \big(R_0^2(1), P(4,1)\big)$
$\big(R_1^4(1), P(5,7)\big)$
$P(9,1): \big(R_1^3(3), P(0,7)\big),\ \big(R_0^2(2), P(1,1)\big),\ \big(R_1^4(2), P(3,1)\big),\ \big(R_\infty^1(1), P(3,7)\big),\ \big(I(1,1), P(4,6)\big)$
$\big(R_0^3(1), P(5,1)\big),\ \big(R_1^1(1), P(6,7)\big)$
$P(10,1): \big(R_1^4(3), P(1,7)\big),\ \big(R_0^3(2), P(2,1)\big),\ \big(R_1^1(2), P(4,1)\big),\ \big(R_\infty^2(1), P(4,7)\big),\ \big(I(0,1), P(5,6)\big)$
$\big(R_0^1(1), P(6,1)\big),\ \big(R_1^2(1), P(7,7)\big)$
$P(11,1): \big(R_1^1(3), P(2,7)\big),\ \big(R_0^1(2), P(3,1)\big),\ \big(R_1^2(2), P(5,1)\big),\ \big(R_\infty^1(1), P(5,7)\big),\ \big(R_0^2(1), P(7,1)\big)$
$\big(R_1^3(1), P(8,7)\big)$
$P(12,1): \big(R_1^2(3), P(3,7)\big),\ \big(R_0^2(2), P(4,1)\big),\ \big(R_1^3(2), P(6,1)\big),\ \big(R_\infty^2(1), P(6,7)\big),\ \big(R_0^3(1), P(8,1)\big)$
$\big(R_1^4(1), P(9,7)\big),\ \big(I(11,1), 2P(0,1)\big)$
$P(n,1): \big(R_1^{(n-11)\bmod 4+1}(3), P(n-9,7)\big),\ \big(R_0^{(n-11)\bmod 3+1}(2), P(n-8,1)\big),\ \big(R_1^{(n-10)\bmod 4+1}(2), P(n-6,1)\big)$
$\big(R_\infty^{(n-11)\bmod 2+1}(1), P(n-6,7)\big),\ \big(R_0^{(n-10)\bmod 3+1}(1), P(n-4,1)\big),\ \big(R_1^{(n-9)\bmod 4+1}(1), P(n-3,7)\big)$
$\big(uI, (u+1)P\big),\ n > 12$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -2$, for $n \geq 0$.

$P(0,2): \big(I(1,1), P(0,3)\big),\ \big(I(1,2), P(0,4)\big)$
$P(1,2): \big(P(1,1), P(0,1)\big),\ \big(I(3,7), P(0,5)\big),\ \big(R_1^4(1), P(0,8)\big),\ \big(I(0,1), P(1,3)\big),\ \big(I(0,2), P(1,4)\big)$
$P(2,2): \big(R_0^1(1), P(0,6)\big),\ \big(P(2,1), P(1,1)\big),\ \big(I(2,7), P(1,5)\big),\ \big(R_1^1(1), P(1,8)\big)$

$P(3,2):\big(R_1^1(2),P(0,2)\big),\ \big(P(5,7),P(0,7)\big),\ \big(R_0^2(1),P(1,6)\big),\ \big(P(3,1),P(2,1)\big),\ \big(I(1,7),P(2,5)\big)$
$\big(R_1^2(1),P(2,8)\big)$

$P(4,2):\big(P(7,1),P(0,1)\big),\ \big(R_1^2(2),P(1,2)\big),\ \big(P(6,7),P(1,7)\big),\ \big(R_0^3(1),P(2,6)\big),\ \big(P(4,1),P(3,1)\big)$
$\big(I(0,7),P(3,5)\big),\ \big(R_1^3(1),P(3,8)\big)$

$P(5,2):\big(P(8,1),P(1,1)\big),\ \big(R_1^3(2),P(2,2)\big),\ \big(P(7,7),P(2,7)\big),\ \big(R_0^1(1),P(3,6)\big),\ \big(P(5,1),P(4,1)\big)$
$\big(R_1^4(1),P(4,8)\big)$

$P(6,2):\big(P(11,7),P(0,7)\big),\ \big(P(9,1),P(2,1)\big),\ \big(R_1^4(2),P(3,2)\big),\ \big(P(8,7),P(3,7)\big),\ \big(R_0^2(1),P(4,6)\big)$
$\big(P(6,1),P(5,1)\big),\ \big(R_1^1(1),P(5,8)\big)$

$P(n,2):\big(P(n+5,7),P(n-6,7)\big),\ \big(P(n+3,1),P(n-4,1)\big),\ \big(R_1^{(n-3)\bmod 4+1}(2),P(n-3,2)\big)$
$\big(P(n+2,7),P(n-3,7)\big),\ \big(R_0^{(n-5)\bmod 3+1}(1),P(n-2,6)\big),\ \big(P(n,1),P(n-1,1)\big)$
$\big(R_1^{(n-6)\bmod 4+1}(1),P(n-1,8)\big),\ n>6$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3)=-3$, for $n\geq 0$.

$P(0,3):\big(I(2,1),P(0,4)\big)$

$P(1,3):\big(P(1,1),P(0,2)\big),\ \big(R_1^3(1),P(0,5)\big),\ \big(P(2,7),P(0,8)\big),\ \big(I(1,1),P(1,4)\big)$

$P(2,3):\big(P(2,2),P(0,1)\big),\ \big(P(4,1),P(0,6)\big),\ \big(P(2,1),P(1,2)\big),\ \big(R_1^4(1),P(1,5)\big),\ \big(P(3,7),P(1,8)\big)$
$\big(I(0,1),P(2,4)\big)$

$P(3,3):\big(P(4,8),P(0,7)\big),\ \big(P(3,2),P(1,1)\big),\ \big(P(5,1),P(1,6)\big),\ \big(P(3,1),P(2,2)\big),\ \big(R_1^1(1),P(2,5)\big)$
$\big(P(4,7),P(2,8)\big)$

$P(4,3):\big(P(5,6),P(0,1)\big),\ \big(P(5,8),P(1,7)\big),\ \big(P(4,2),P(2,1)\big),\ \big(P(6,1),P(2,6)\big),\ \big(P(4,1),P(3,2)\big)$
$\big(R_1^2(1),P(3,5)\big),\ \big(P(5,7),P(3,8)\big)$

$P(n,3):\big(P(n+1,6),P(n-4,1)\big),\ \big(P(n+1,8),P(n-3,7)\big),\ \big(P(n,2),P(n-2,1)\big)$
$\big(P(n+2,1),P(n-2,6)\big),\ \big(P(n,1),P(n-1,2)\big),\ \big(R_1^{(n-3)\bmod 4+1}(1),P(n-1,5)\big)$
$\big(P(n+1,7),P(n-1,8)\big),\ n>4$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4)=-4$, for $n\geq 0$.

$P(0,4):-$

$P(1,4):\big(P(1,1),P(0,3)\big),\ \big(P(1,7),P(0,5)\big),\ \big(P(1,8),P(0,8)\big)$

$P(2,4):\big(P(2,2),P(0,2)\big),\ \big(P(2,6),P(0,6)\big),\ \big(P(2,1),P(1,3)\big),\ \big(P(2,7),P(1,5)\big),\ \big(P(2,8),P(1,8)\big)$

$P(3,4):\big(P(3,3),P(0,1)\big),\ \big(P(3,5),P(0,7)\big),\ \big(P(3,2),P(1,2)\big),\ \big(P(3,6),P(1,6)\big),\ \big(P(3,1),P(2,3)\big)$

$\big(P(3,7), P(2,5)\big),\ \big(P(3,8), P(2,8)\big)$
$P(n,4) : \big(P(n,3), P(n-3,1)\big),\ \big(P(n,5), P(n-3,7)\big),\ \big(P(n,2), P(n-2,2)\big)$
$\big(P(n,6), P(n-2,6)\big),\ \big(P(n,1), P(n-1,3)\big),\ \big(P(n,7), P(n-1,5)\big)$
$\big(P(n,8), P(n-1,8)\big),\ n > 3$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -3$, for $n \geq 0$.

$P(0,5) : \big(I(2,7), P(0,4)\big)$
$P(1,5) : \big(R_1^1(1), P(0,3)\big),\ \big(P(1,7), P(0,6)\big),\ \big(P(2,1), P(0,8)\big),\ \big(I(1,7), P(1,4)\big)$
$P(2,5) : \big(P(4,7), P(0,2)\big),\ \big(P(2,6), P(0,7)\big),\ \big(R_1^2(1), P(1,3)\big),\ \big(P(2,7), P(1,6)\big),\ \big(P(3,1), P(1,8)\big)$
$\big(I(0,7), P(2,4)\big)$
$P(3,5) : \big(P(4,8), P(0,1)\big),\ \big(P(5,7), P(1,2)\big),\ \big(P(3,6), P(1,7)\big),\ \big(R_1^3(1), P(2,3)\big),\ \big(P(3,7), P(2,6)\big)$
$\big(P(4,1), P(2,8)\big)$
$P(4,5) : \big(P(5,2), P(0,7)\big),\ \big(P(5,8), P(1,1)\big),\ \big(P(6,7), P(2,2)\big),\ \big(P(4,6), P(2,7)\big),\ \big(R_1^4(1), P(3,3)\big)$
$\big(P(4,7), P(3,6)\big),\ \big(P(5,1), P(3,8)\big)$
$P(n,5) : \big(P(n+1,2), P(n-4,7)\big),\ \big(P(n+1,8), P(n-3,1)\big),\ \big(P(n+2,7), P(n-2,2)\big)$
$\big(P(n,6), P(n-2,7)\big),\ \big(R_1^{(n-1) \bmod 4+1}(1), P(n-1,3)\big),\ \big(P(n,7), P(n-1,6)\big)$
$\big(P(n+1,1), P(n-1,8)\big),\ n > 4$

Modules of the form $P(n,6)$

Defect: $\partial P(n,6) = -2$, for $n \geq 0$.

$P(0,6) : \big(I(1,6), P(0,4)\big),\ \big(I(1,7), P(0,5)\big)$
$P(1,6) : \big(I(3,1), P(0,3)\big),\ \big(P(1,7), P(0,7)\big),\ \big(R_1^2(1), P(0,8)\big),\ \big(I(0,6), P(1,4)\big),\ \big(I(0,7), P(1,5)\big)$
$P(2,6) : \big(R_0^1(1), P(0,2)\big),\ \big(I(2,1), P(1,3)\big),\ \big(P(2,7), P(1,7)\big),\ \big(R_1^3(1), P(1,8)\big)$
$P(3,6) : \big(P(5,1), P(0,1)\big),\ \big(R_1^3(2), P(0,6)\big),\ \big(R_0^2(1), P(1,2)\big),\ \big(I(1,1), P(2,3)\big),\ \big(P(3,7), P(2,7)\big)$
$\big(R_1^4(1), P(2,8)\big)$
$P(4,6) : \big(P(7,7), P(0,7)\big),\ \big(P(6,1), P(1,1)\big),\ \big(R_1^4(2), P(1,6)\big),\ \big(R_0^3(1), P(2,2)\big),\ \big(I(0,1), P(3,3)\big)$
$\big(P(4,7), P(3,7)\big),\ \big(R_1^1(1), P(3,8)\big)$
$P(5,6) : \big(P(8,7), P(1,7)\big),\ \big(P(7,1), P(2,1)\big),\ \big(R_1^1(2), P(2,6)\big),\ \big(R_0^1(1), P(3,2)\big),\ \big(P(5,7), P(4,7)\big)$
$\big(R_1^2(1), P(4,8)\big)$
$P(6,6) : \big(P(11,1), P(0,1)\big),\ \big(P(9,7), P(2,7)\big),\ \big(P(8,1), P(3,1)\big),\ \big(R_1^2(2), P(3,6)\big),\ \big(R_0^2(1), P(4,2)\big)$
$\big(P(6,7), P(5,7)\big),\ \big(R_1^3(1), P(5,8)\big)$

$P(n,6) : \big(P(n+5,1), P(n-6,1)\big),\ \big(P(n+3,7), P(n-4,7)\big),\ \big(P(n+2,1), P(n-3,1)\big)$
$\big(R_1^{(n-5) \bmod 4+1}(2), P(n-3,6)\big),\ \big(R_0^{(n-5) \bmod 3+1}(1), P(n-2,2)\big),\ \big(P(n,7), P(n-1,7)\big)$
$\big(R_1^{(n-4) \bmod 4+1}(1), P(n-1,8)\big),\ n > 6$

Modules of the form $P(n,7)$

Defect: $\partial P(n,7) = -1$, for $n \geq 0$.

$P(0,7) : \big(I(0,5), P(0,4)\big),\ \big(I(0,6), P(0,5)\big),\ \big(I(0,7), P(0,6)\big)$
$P(1,7) : \big(I(0,8), P(0,3)\big),\ \big(I(2,1), P(0,8)\big)$
$P(2,7) : \big(I(2,7), P(0,2)\big),\ \big(I(1,2), P(0,5)\big),\ \big(I(1,1), P(1,8)\big)$
$P(3,7) : \big(R_1^1(1), P(0,1)\big),\ \big(I(3,7), P(0,6)\big),\ \big(I(4,1), P(0,8)\big),\ \big(I(1,7), P(1,2)\big),\ \big(I(0,2), P(1,5)\big)$
$\big(I(0,1), P(2,8)\big)$
$P(4,7) : \big(R_0^1(1), P(0,7)\big),\ \big(R_1^2(1), P(1,1)\big),\ \big(I(2,7), P(1,6)\big),\ \big(I(3,1), P(1,8)\big),\ \big(I(0,7), P(2,2)\big)$
$P(5,7) : \big(I(5,7), P(0,2)\big),\ \big(R_0^2(1), P(1,7)\big),\ \big(R_1^3(1), P(2,1)\big),\ \big(I(1,7), P(2,6)\big),\ \big(I(2,1), P(2,8)\big)$
$P(6,7) : \big(R_\infty^1(1), P(0,1)\big),\ \big(R_1^3(2), P(0,7)\big),\ \big(I(4,7), P(1,2)\big),\ \big(R_0^3(1), P(2,7)\big),\ \big(R_1^4(1), P(3,1)\big)$
$\big(I(0,7), P(3,6)\big),\ \big(I(1,1), P(3,8)\big)$
$P(7,7) : \big(R_\infty^2(1), P(1,1)\big),\ \big(R_1^4(2), P(1,7)\big),\ \big(I(3,7), P(2,2)\big),\ \big(R_0^1(1), P(3,7)\big),\ \big(R_1^1(1), P(4,1)\big)$
$\big(I(0,1), P(4,8)\big)$
$P(8,7) : \big(R_0^1(2), P(0,7)\big),\ \big(R_\infty^1(1), P(2,1)\big),\ \big(R_1^1(2), P(2,7)\big),\ \big(I(2,7), P(3,2)\big),\ \big(R_0^2(1), P(4,7)\big)$
$\big(R_1^2(1), P(5,1)\big)$
$P(9,7) : \big(R_1^1(3), P(0,1)\big),\ \big(R_0^2(2), P(1,7)\big),\ \big(R_\infty^2(1), P(3,1)\big),\ \big(R_1^2(2), P(3,7)\big),\ \big(I(1,7), P(4,2)\big)$
$\big(R_0^3(1), P(5,7)\big),\ \big(R_1^3(1), P(6,1)\big)$
$P(10,7) : \big(R_1^2(3), P(1,1)\big),\ \big(R_0^3(2), P(2,7)\big),\ \big(R_\infty^1(1), P(4,1)\big),\ \big(R_1^3(2), P(4,7)\big),\ \big(I(0,7), P(5,2)\big)$
$\big(R_0^1(1), P(6,7)\big),\ \big(R_1^4(1), P(7,1)\big)$
$P(11,7) : \big(R_1^3(3), P(2,1)\big),\ \big(R_0^1(2), P(3,7)\big),\ \big(R_\infty^2(1), P(5,1)\big),\ \big(R_1^4(2), P(5,7)\big),\ \big(R_0^2(1), P(7,7)\big)$
$\big(R_1^1(1), P(8,1)\big)$
$P(12,7) : \big(R_1^4(3), P(3,1)\big),\ \big(R_0^2(2), P(4,7)\big),\ \big(R_\infty^1(1), P(6,1)\big),\ \big(R_1^1(2), P(6,7)\big),\ \big(R_0^3(1), P(8,7)\big)$
$\big(R_1^2(1), P(9,1)\big),\ \big(I(11,7), 2P(0,7)\big)$
$P(n,7) : \big(R_1^{(n-9) \bmod 4+1}(3), P(n-9,1)\big),\ \big(R_0^{(n-11) \bmod 3+1}(2), P(n-8,7)\big),\ \big(R_\infty^{(n-12) \bmod 2+1}(1), P(n-6,1)\big)$
$\big(R_1^{(n-12) \bmod 4+1}(2), P(n-6,7)\big),\ \big(R_0^{(n-10) \bmod 3+1}(1), P(n-4,7)\big),\ \big(R_1^{(n-11) \bmod 4+1}(1), P(n-3,1)\big)$
$\big(uI, (u+1)P\big),\ n > 12$

Modules of the form $P(n,8)$

Defect: $\partial P(n,8) = -2$, for $n \geq 0$.

$P(0,8): \big(I(0,8), P(0,4)\big)$

$P(1,8): \big(I(2,7), P(0,3)\big),\ \big(I(2,1), P(0,5)\big)$

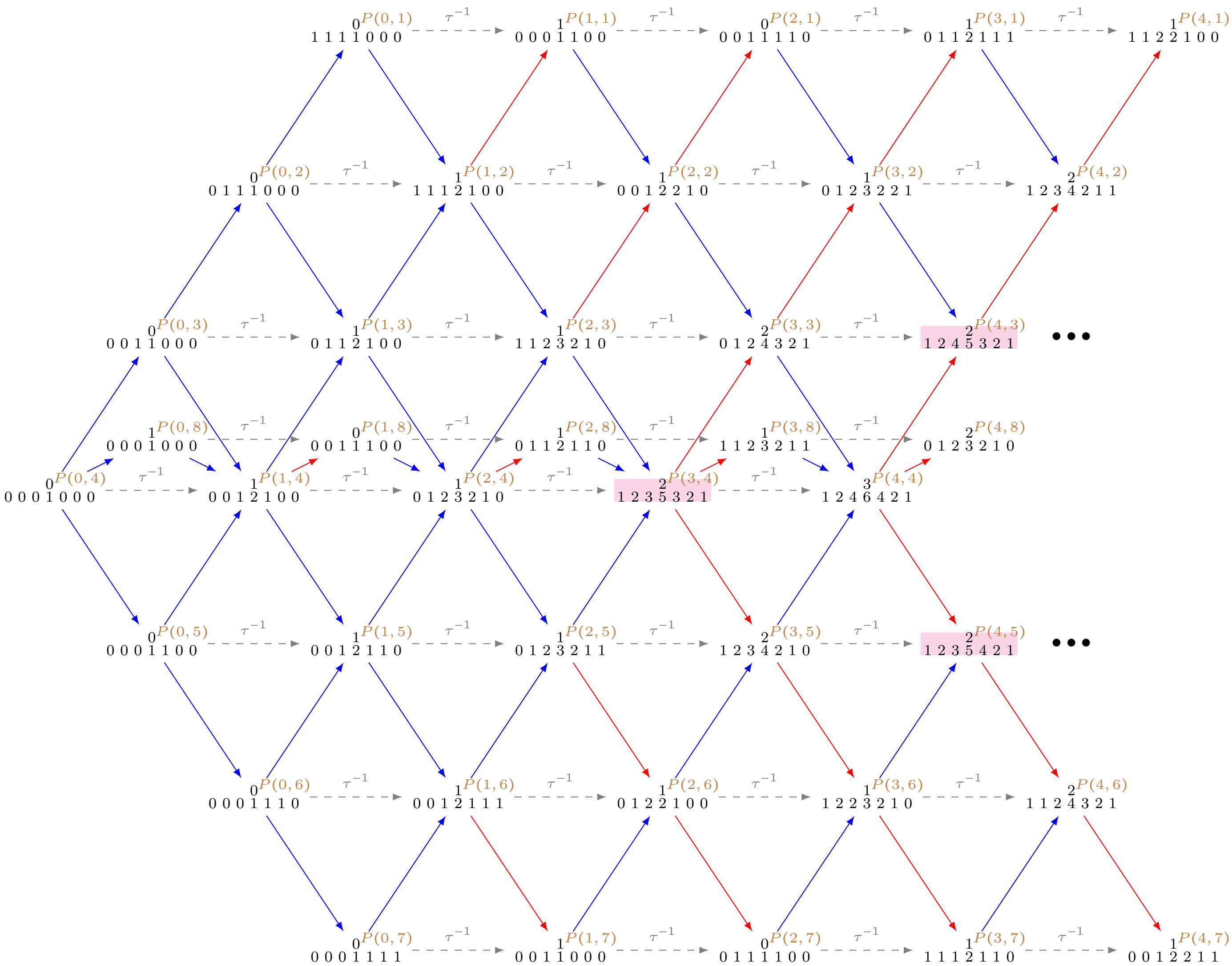
$P(2,8): \big(R_1^1(1), P(0,2)\big),\ \big(R_1^3(1), P(0,6)\big),\ \big(R_0^2(1), P(0,8)\big),\ \big(I(1,7), P(1,3)\big),\ \big(I(1,1), P(1,5)\big)$

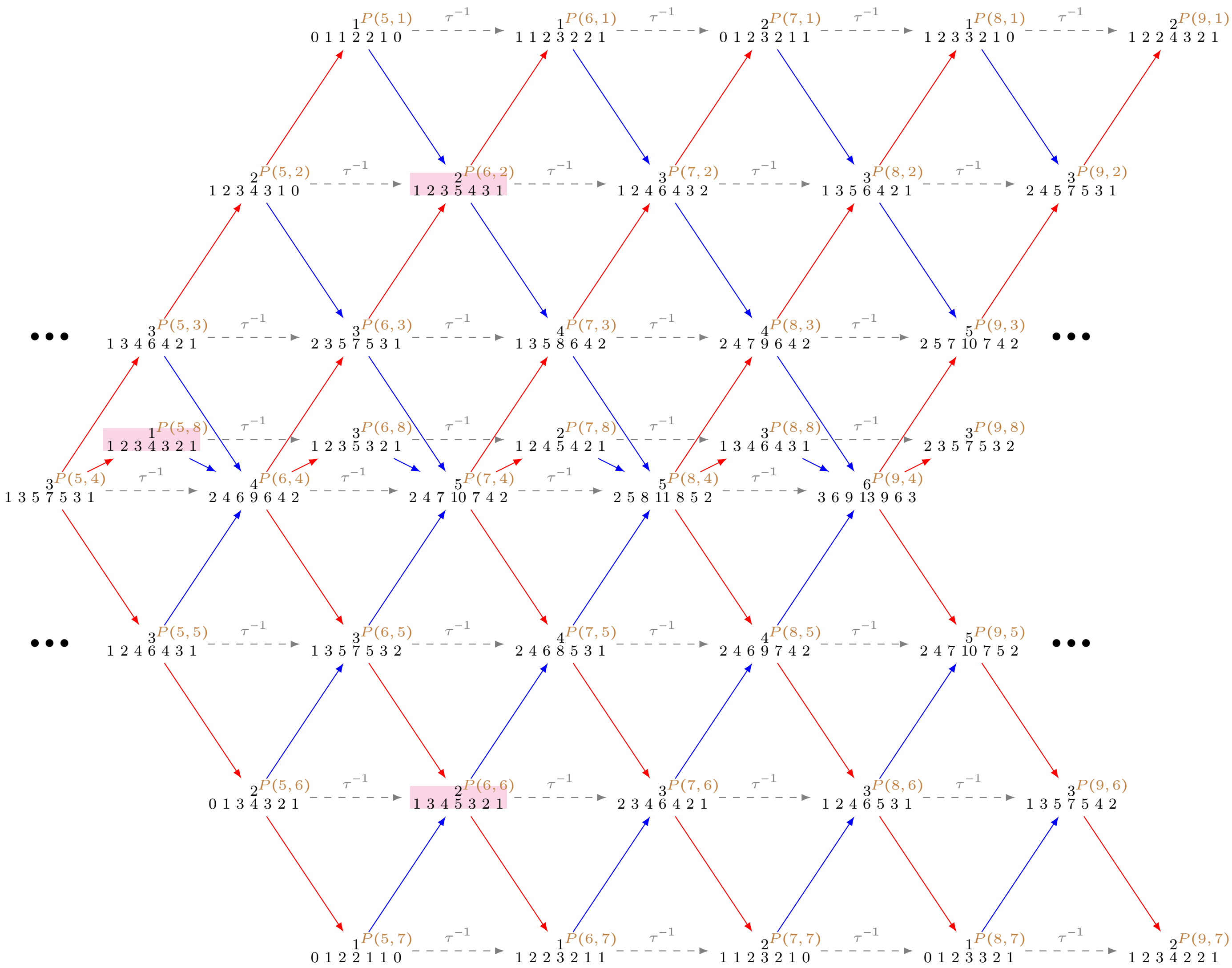
$P(3,8): \big(P(4,7), P(0,1)\big),\ \big(P(4,1), P(0,7)\big),\ \big(R_1^2(1), P(1,2)\big),\ \big(R_1^4(1), P(1,6)\big),\ \big(R_0^3(1), P(1,8)\big)$
$\big(I(0,7), P(2,3)\big),\ \big(I(0,1), P(2,5)\big)$

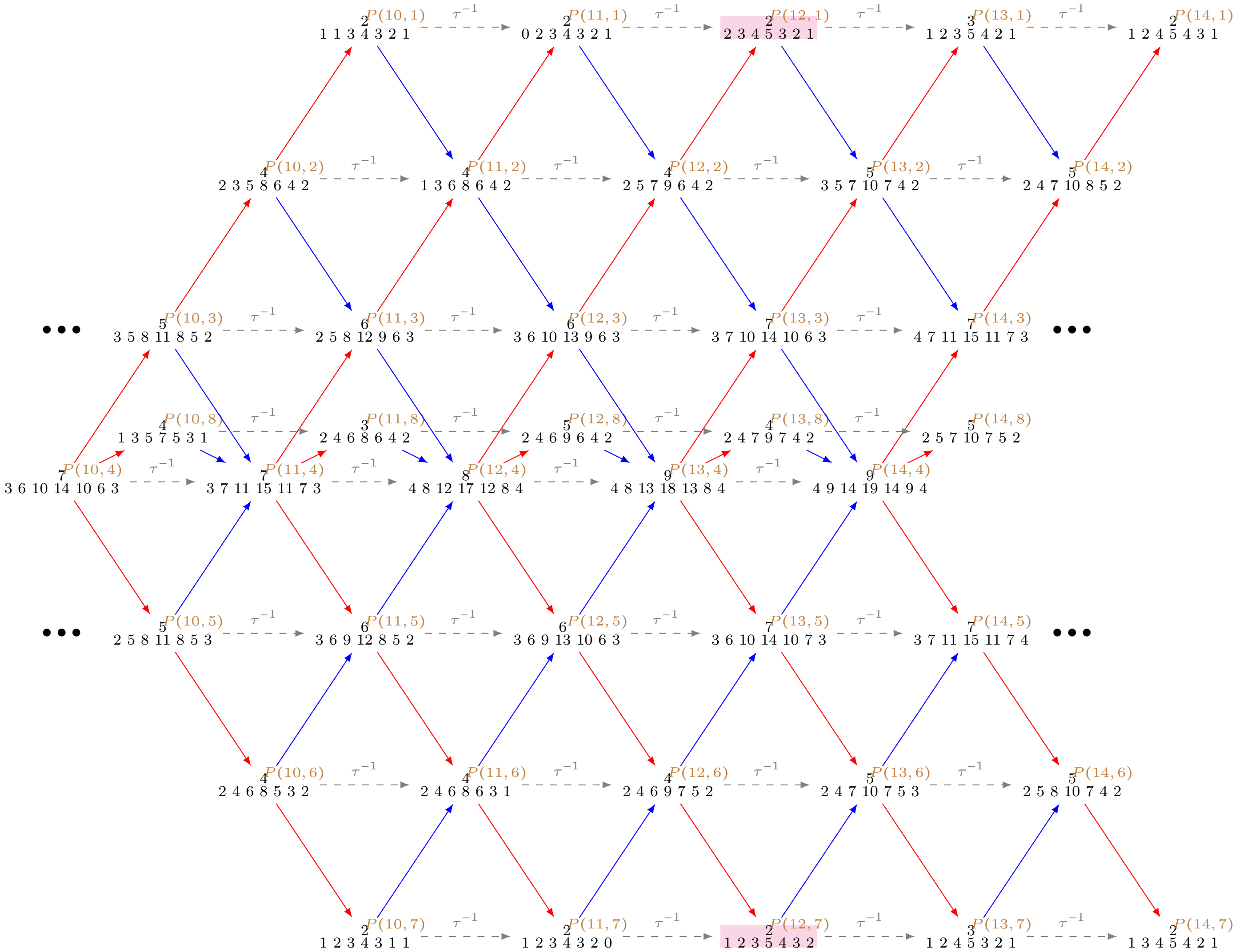
$P(4,8): \big(P(5,7), P(1,1)\big),\ \big(P(5,1), P(1,7)\big),\ \big(R_1^3(1), P(2,2)\big),\ \big(R_1^1(1), P(2,6)\big),\ \big(R_0^1(1), P(2,8)\big)$

$P(5,8): \big(P(8,7), P(0,1)\big),\ \big(P(8,1), P(0,7)\big),\ \big(P(6,7), P(2,1)\big),\ \big(P(6,1), P(2,7)\big),\ \big(R_1^4(1), P(3,2)\big)$
$\big(R_1^2(1), P(3,6)\big),\ \big(R_0^2(1), P(3,8)\big)$

$P(n,8): \big(P(n+3,7), P(n-5,1)\big),\ \big(P(n+3,1), P(n-5,7)\big),\ \big(P(n+1,7), P(n-3,1)\big)$
$\big(P(n+1,1), P(n-3,7)\big),\ \big(R_1^{(n-2) \bmod 4+1}(1), P(n-2,2)\big),\ \big(R_1^{(n-4) \bmod 4+1}(1), P(n-2,6)\big)$
$\big(R_0^{(n-4) \bmod 3+1}(1), P(n-2,8)\big),\ n > 5$






P(10,1)
P(11,1)
P(12,1)
P(13,1)
P(14,1)
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P(14,7)
τ^{-1}

Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 1$, for $n \geq 0$.

$I(0,1)$: –

$I(1,1)$: –

$I(2,1)$: –

$I(3,1) : \big(I(0,5), P(0,8)\big),\ \big(I(0,6), P(1,1)\big),\ \big(I(0,7), R_1^1(1)\big),\ \big(I(0,8), P(0,7)\big)$

$I(4,1) : \big(I(0,1), R_0^2(1)\big),\ \big(I(0,2), P(2,1)\big),\ \big(I(0,3), P(0,6)\big),\ \big(I(1,6), P(0,1)\big),\ \big(I(1,7), R_1^4(1)\big)$

$I(5,1) : \big(I(0,8), P(2,7)\big),\ \big(I(1,1), R_0^1(1)\big),\ \big(I(1,2), P(1,1)\big),\ \big(I(2,7), R_1^3(1)\big)$

$I(6,1) : \big(I(0,1), R_1^2(2)\big),\ \big(I(0,6), P(4,1)\big),\ \big(I(0,7), R_\infty^2(1)\big),\ \big(I(1,8), P(1,7)\big),\ \big(I(2,1), R_0^3(1)\big)$
$\big(I(2,2), P(0,1)\big),\ \big(I(3,7), R_1^2(1)\big)$

$I(7,1) : \big(I(1,1), R_1^1(2)\big),\ \big(I(1,6), P(3,1)\big),\ \big(I(1,7), R_\infty^1(1)\big),\ \big(I(2,8), P(0,7)\big),\ \big(I(3,1), R_0^2(1)\big)$
$\big(I(4,7), R_1^1(1)\big)$

$I(8,1) : \big(I(0,1), R_0^1(2)\big),\ \big(I(2,1), R_1^4(2)\big),\ \big(I(2,6), P(2,1)\big),\ \big(I(2,7), R_\infty^2(1)\big),\ \big(I(4,1), R_0^1(1)\big)$
$\big(I(5,7), R_1^4(1)\big)$

$I(9,1) : \big(I(0,7), R_1^3(3)\big),\ \big(I(1,1), R_0^3(2)\big),\ \big(I(3,1), R_1^3(2)\big),\ \big(I(3,6), P(1,1)\big),\ \big(I(3,7), R_\infty^1(1)\big)$
$\big(I(5,1), R_0^3(1)\big),\ \big(I(6,7), R_1^3(1)\big)$

$I(10,1) : \big(I(1,7), R_1^2(3)\big),\ \big(I(2,1), R_0^2(2)\big),\ \big(I(4,1), R_1^2(2)\big),\ \big(I(4,6), P(0,1)\big),\ \big(I(4,7), R_\infty^2(1)\big)$
$\big(I(6,1), R_0^2(1)\big),\ \big(I(7,7), R_1^2(1)\big)$

$I(11,1) : \big(I(2,7), R_1^1(3)\big),\ \big(I(3,1), R_0^1(2)\big),\ \big(I(5,1), R_1^1(2)\big),\ \big(I(5,7), R_\infty^1(1)\big),\ \big(I(7,1), R_0^1(1)\big)$
$\big(I(8,7), R_1^1(1)\big)$

$I(12,1) : \big(I(3,7), R_1^4(3)\big),\ \big(I(4,1), R_0^3(2)\big),\ \big(I(6,1), R_1^4(2)\big),\ \big(I(6,7), R_\infty^2(1)\big),\ \big(I(8,1), R_0^3(1)\big)$
$\big(I(9,7), R_1^4(1)\big),\ \big(2I(0,1), P(11,1)\big)$

$I(n,1) : \big(I(n-9,7), R_1^{(-n+15) \bmod 4+1}(3)\big),\ \big(I(n-8,1), R_0^{(-n+14) \bmod 3+1}(2)\big),\ \big(I(n-6,1), R_1^{(-n+15) \bmod 4+1}(2)\big)$
$\big(I(n-6,7), R_\infty^{(-n+13) \bmod 2+1}(1)\big),\ \big(I(n-4,1), R_0^{(-n+14) \bmod 3+1}(1)\big),\ \big(I(n-3,7), R_1^{(-n+15) \bmod 4+1}(1)\big)$
$\big((v+1)I, vP\big),\ n > 12$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 2$, for $n \geq 0$.

$I(0,2) : \big(I(0,1), I(1,1)\big)$

$I(1,2) : \big(I(1,1), I(2,1)\big)$

$I(2,2) : \big(I(0,5), P(1,7)\big),\ \big(I(0,6), R_0^1(1)\big),\ \big(I(0,7), I(5,7)\big),\ \big(I(0,8), R_1^2(1)\big),\ \big(I(2,1), I(3,1)\big)$

$I(3,2) : \big(I(0,1), I(7,1)\big),\ \big(I(0,2), R_1^1(2)\big),\ \big(I(1,5), P(0,7)\big),\ \big(I(1,6), R_0^3(1)\big),\ \big(I(1,7), I(6,7)\big)$

$\big(I(1,8), R_1^1(1)\big),\ \big(I(3,1), I(4,1)\big)$

$I(4,2) : \big(I(1,1), I(8,1)\big),\ \big(I(1,2), R_1^4(2)\big),\ \big(I(2,6), R_0^2(1)\big),\ \big(I(2,7), I(7,7)\big),\ \big(I(2,8), R_1^4(1)\big)$
$\big(I(4,1), I(5,1)\big)$

$I(5,2) : \big(I(0,7), I(11,7)\big),\ \big(I(2,1), I(9,1)\big),\ \big(I(2,2), R_1^3(2)\big),\ \big(I(3,6), R_0^1(1)\big),\ \big(I(3,7), I(8,7)\big)$
$\big(I(3,8), R_1^3(1)\big),\ \big(I(5,1), I(6,1)\big)$

$I(n,2) : \big(I(n-5,7), I(n+6,7)\big),\ \big(I(n-3,1), I(n+4,1)\big),\ \big(I(n-3,2), R_1^{(-n+7) \bmod 4+1}(2)\big)$
$\big(I(n-2,6), R_0^{(-n+5) \bmod 3+1}(1)\big),\ \big(I(n-2,7), I(n+3,7)\big),\ \big(I(n-2,8), R_1^{(-n+7) \bmod 4+1}(1)\big)$
$\big(I(n,1), I(n+1,1)\big),\ n > 5$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 3$, for $n \geq 0$.

$I(0,3) : \big(I(0,1), I(1,2)\big),\ \big(I(0,2), I(2,1)\big)$

$I(1,3) : \big(I(0,5), R_1^3(1)\big),\ \big(I(0,6), I(5,1)\big),\ \big(I(0,7), I(2,8)\big),\ \big(I(0,8), I(4,7)\big),\ \big(I(1,1), I(2,2)\big)$
$\big(I(1,2), I(3,1)\big)$

$I(2,3) : \big(I(0,1), I(4,6)\big),\ \big(I(1,5), R_1^2(1)\big),\ \big(I(1,6), I(6,1)\big),\ \big(I(1,7), I(3,8)\big),\ \big(I(1,8), I(5,7)\big)$
$\big(I(2,1), I(3,2)\big),\ \big(I(2,2), I(4,1)\big)$

$I(n,3) : \big(I(n-2,1), I(n+2,6)\big),\ \big(I(n-1,5), R_1^{(-n+3) \bmod 4+1}(1)\big),\ \big(I(n-1,6), I(n+4,1)\big)$
$\big(I(n-1,7), I(n+1,8)\big),\ \big(I(n-1,8), I(n+3,7)\big),\ \big(I(n,1), I(n+1,2)\big)$
$\big(I(n,2), I(n+2,1)\big),\ n > 2$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 4$, for $n \geq 0$.

$I(0,4) : \big(I(0,1), I(1,3)\big),\ \big(I(0,2), I(2,2)\big),\ \big(I(0,3), I(3,1)\big),\ \big(I(0,5), I(3,7)\big),\ \big(I(0,6), I(2,6)\big)$
$\big(I(0,7), I(1,5)\big),\ \big(I(0,8), I(1,8)\big)$

$I(n,4) : \big(I(n,1), I(n+1,3)\big),\ \big(I(n,2), I(n+2,2)\big),\ \big(I(n,3), I(n+3,1)\big)$
$\big(I(n,5), I(n+3,7)\big),\ \big(I(n,6), I(n+2,6)\big),\ \big(I(n,7), I(n+1,5)\big)$
$\big(I(n,8), I(n+1,8)\big),\ n > 0$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 3$, for $n \geq 0$.

$I(0,5) : \big(I(0,6), I(2,7)\big),\ \big(I(0,7), I(1,6)\big)$

$I(1,5) : \big(I(0,1), I(2,8)\big),\ \big(I(0,2), I(5,7)\big),\ \big(I(0,3), R_1^1(1)\big),\ \big(I(0,8), I(4,1)\big),\ \big(I(1,6), I(3,7)\big)$
$\big(I(1,7), I(2,6)\big)$

$I(2,5) : \big(I(0,7), I(4,2)\big),\ \big(I(1,1), I(3,8)\big),\ \big(I(1,2), I(6,7)\big),\ \big(I(1,3), R_1^4(1)\big),\ \big(I(1,8), I(5,1)\big)$
$\big(I(2,6), I(4,7)\big),\ \big(I(2,7), I(3,6)\big)$

$I(n,5) : \big(I(n-2,7), I(n+2,2)\big),\ \big(I(n-1,1), I(n+1,8)\big),\ \big(I(n-1,2), I(n+4,7)\big)$
$\big(I(n-1,3), R_1^{(-n+5) \bmod 4+1}(1)\big),\ \big(I(n-1,8), I(n+3,1)\big),\ \big(I(n,6), I(n+2,7)\big)$
$\big(I(n,7), I(n+1,6)\big),\ n > 2$

Modules of the form $I(n,6)$

Defect: $\partial I(n,6) = 2$, for $n \geq 0$.

$I(0,6) : \big(I(0,7), I(1,7)\big)$

$I(1,6) : \big(I(1,7), I(2,7)\big)$

$I(2,6) : \big(I(0,1), I(5,1)\big),\ \big(I(0,2), R_0^1(1)\big),\ \big(I(0,3), P(1,1)\big),\ \big(I(0,8), R_1^4(1)\big),\ \big(I(2,7), I(3,7)\big)$

$I(3,6) : \big(I(0,6), R_1^3(2)\big),\ \big(I(0,7), I(7,7)\big),\ \big(I(1,1), I(6,1)\big),\ \big(I(1,2), R_0^3(1)\big),\ \big(I(1,3), P(0,1)\big)$
$\big(I(1,8), R_1^3(1)\big),\ \big(I(3,7), I(4,7)\big)$

$I(4,6) : \big(I(1,6), R_1^2(2)\big),\ \big(I(1,7), I(8,7)\big),\ \big(I(2,1), I(7,1)\big),\ \big(I(2,2), R_0^2(1)\big),\ \big(I(2,8), R_1^2(1)\big)$
$\big(I(4,7), I(5,7)\big)$

$I(5,6) : \big(I(0,1), I(11,1)\big),\ \big(I(2,6), R_1^1(2)\big),\ \big(I(2,7), I(9,7)\big),\ \big(I(3,1), I(8,1)\big),\ \big(I(3,2), R_0^1(1)\big)$
$\big(I(3,8), R_1^1(1)\big),\ \big(I(5,7), I(6,7)\big)$

$I(n,6) : \big(I(n-5,1), I(n+6,1)\big),\ \big(I(n-3,6), R_1^{(-n+5) \bmod 4+1}(2)\big),\ \big(I(n-3,7), I(n+4,7)\big)$
$\big(I(n-2,1), I(n+3,1)\big),\ \big(I(n-2,2), R_0^{(-n+5) \bmod 3+1}(1)\big),\ \big(I(n-2,8), R_1^{(-n+5) \bmod 4+1}(1)\big)$
$\big(I(n,7), I(n+1,7)\big),\ n > 5$

Modules of the form $I(n,7)$

Defect: $\partial I(n,7) = 1$, for $n \geq 0$.

$I(0,7) : -$

$I(1,7) : -$

$I(2,7) : -$

$I(3,7) : \big(I(0,1), R_1^3(1)\big),\ \big(I(0,2), P(1,7)\big),\ \big(I(0,3), P(0,8)\big),\ \big(I(0,8), P(0,1)\big)$

$I(4,7) : \big(I(0,5), P(0,2)\big),\ \big(I(0,6), P(2,7)\big),\ \big(I(0,7), R_0^2(1)\big),\ \big(I(1,1), R_1^2(1)\big),\ \big(I(1,2), P(0,7)\big)$

$I(5,7) : \big(I(0,8), P(2,1)\big),\ \big(I(1,6), P(1,7)\big),\ \big(I(1,7), R_0^1(1)\big),\ \big(I(2,1), R_1^1(1)\big)$

$I(6,7) : \big(I(0,1), R_\infty^1(1)\big),\ \big(I(0,2), P(4,7)\big),\ \big(I(0,7), R_1^4(2)\big),\ \big(I(1,8), P(1,1)\big),\ \big(I(2,6), P(0,7)\big)$

$\big(I(2,7), R_0^3(1)\big),\ \big(I(3,1), R_1^4(1)\big)$

$I(7,7): \big(I(1,1), R_\infty^2(1)\big),\ \big(I(1,2), P(3,7)\big),\ \big(I(1,7), R_1^3(2)\big),\ \big(I(2,8), P(0,1)\big),\ \big(I(3,7), R_0^2(1)\big)$
$\big(I(4,1), R_1^3(1)\big)$

$I(8,7): \big(I(0,7), R_0^1(2)\big),\ \big(I(2,1), R_\infty^1(1)\big),\ \big(I(2,2), P(2,7)\big),\ \big(I(2,7), R_1^2(2)\big),\ \big(I(4,7), R_0^1(1)\big)$
$\big(I(5,1), R_1^2(1)\big)$

$I(9,7): \big(I(0,1), R_1^1(3)\big),\ \big(I(1,7), R_0^3(2)\big),\ \big(I(3,1), R_\infty^2(1)\big),\ \big(I(3,2), P(1,7)\big),\ \big(I(3,7), R_1^1(2)\big)$
$\big(I(5,7), R_0^3(1)\big),\ \big(I(6,1), R_1^1(1)\big)$

$I(10,7): \big(I(1,1), R_1^4(3)\big),\ \big(I(2,7), R_0^2(2)\big),\ \big(I(4,1), R_\infty^1(1)\big),\ \big(I(4,2), P(0,7)\big),\ \big(I(4,7), R_1^4(2)\big)$
$\big(I(6,7), R_0^2(1)\big),\ \big(I(7,1), R_1^4(1)\big)$

$I(11,7): \big(I(2,1), R_1^3(3)\big),\ \big(I(3,7), R_0^1(2)\big),\ \big(I(5,1), R_\infty^2(1)\big),\ \big(I(5,7), R_1^3(2)\big),\ \big(I(7,7), R_0^1(1)\big)$
$\big(I(8,1), R_1^3(1)\big)$

$I(12,7): \big(I(3,1), R_1^2(3)\big),\ \big(I(4,7), R_0^3(2)\big),\ \big(I(6,1), R_\infty^1(1)\big),\ \big(I(6,7), R_1^2(2)\big),\ \big(I(8,7), R_0^3(1)\big)$
$\big(I(9,1), R_1^2(1)\big),\ \big(2I(0,7), P(11,7)\big)$

$I(n,7): \big(I(n-9,1), R_1^{(-n+13) \bmod 4+1}(3)\big),\ \big(I(n-8,7), R_0^{(-n+14) \bmod 3+1}(2)\big),\ \big(I(n-6,1), R_\infty^{(-n+12) \bmod 2+1}(1)\big)$
$\big(I(n-6,7), R_1^{(-n+13) \bmod 4+1}(2)\big),\ \big(I(n-4,7), R_0^{(-n+14) \bmod 3+1}(1)\big),\ \big(I(n-3,1), R_1^{(-n+13) \bmod 4+1}(1)\big)$
$\big((v+1)I, vP\big),\ n > 12$

Modules of the form $I(n,8)$

Defect: $\partial I(n,8) = 2$, for $n \geq 0$.

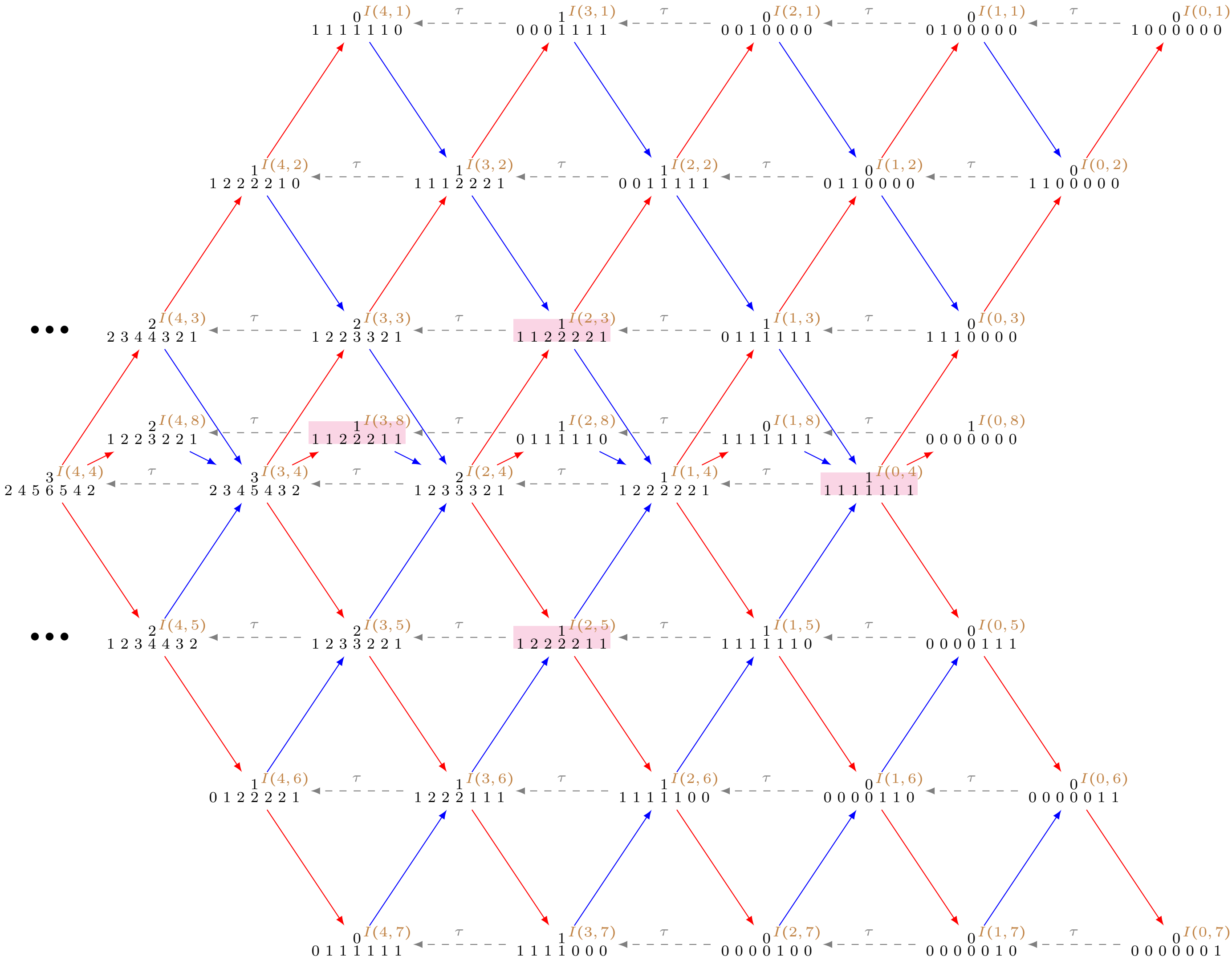
$I(0,8): -$

$I(1,8): \big(I(0,1), I(4,7)\big),\ \big(I(0,2), R_1^2(1)\big),\ \big(I(0,3), P(0,7)\big),\ \big(I(0,5), P(0,1)\big),\ \big(I(0,6), R_1^4(1)\big)$
$\big(I(0,7), I(4,1)\big)$

$I(2,8): \big(I(0,8), R_0^2(1)\big),\ \big(I(1,1), I(5,7)\big),\ \big(I(1,2), R_1^1(1)\big),\ \big(I(1,6), R_1^3(1)\big),\ \big(I(1,7), I(5,1)\big)$

$I(3,8): \big(I(0,1), I(8,7)\big),\ \big(I(0,7), I(8,1)\big),\ \big(I(1,8), R_0^1(1)\big),\ \big(I(2,1), I(6,7)\big),\ \big(I(2,2), R_1^4(1)\big)$
$\big(I(2,6), R_1^2(1)\big),\ \big(I(2,7), I(6,1)\big)$

$I(n,8): \big(I(n-3,1), I(n+5,7)\big),\ \big(I(n-3,7), I(n+5,1)\big),\ \big(I(n-2,8), R_0^{(-n+3) \bmod 3+1}(1)\big)$
$\big(I(n-1,1), I(n+3,7)\big),\ \big(I(n-1,2), R_1^{(-n+6) \bmod 4+1}(1)\big),\ \big(I(n-1,6), R_1^{(-n+4) \bmod 4+1}(1)\big)$
$\big(I(n-1,7), I(n+3,1)\big),\ n > 3$


I(4,1)
I(3,1)
I(2,1)
I(1,1)
I(0,1)
I(4,2)
I(3,2)
I(2,2)
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I(0,2)
I(4,3)
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I(0,3)
I(4,8)
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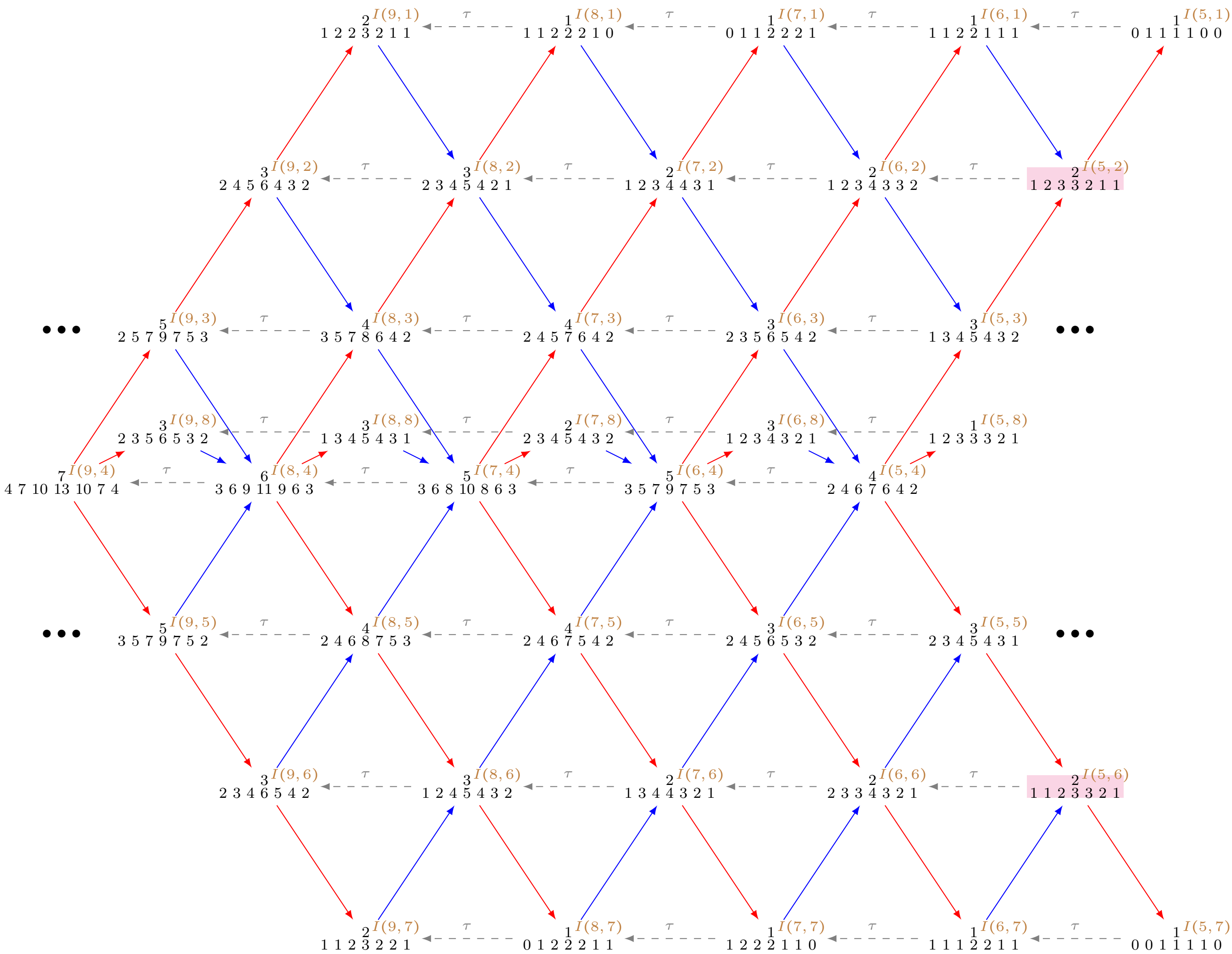
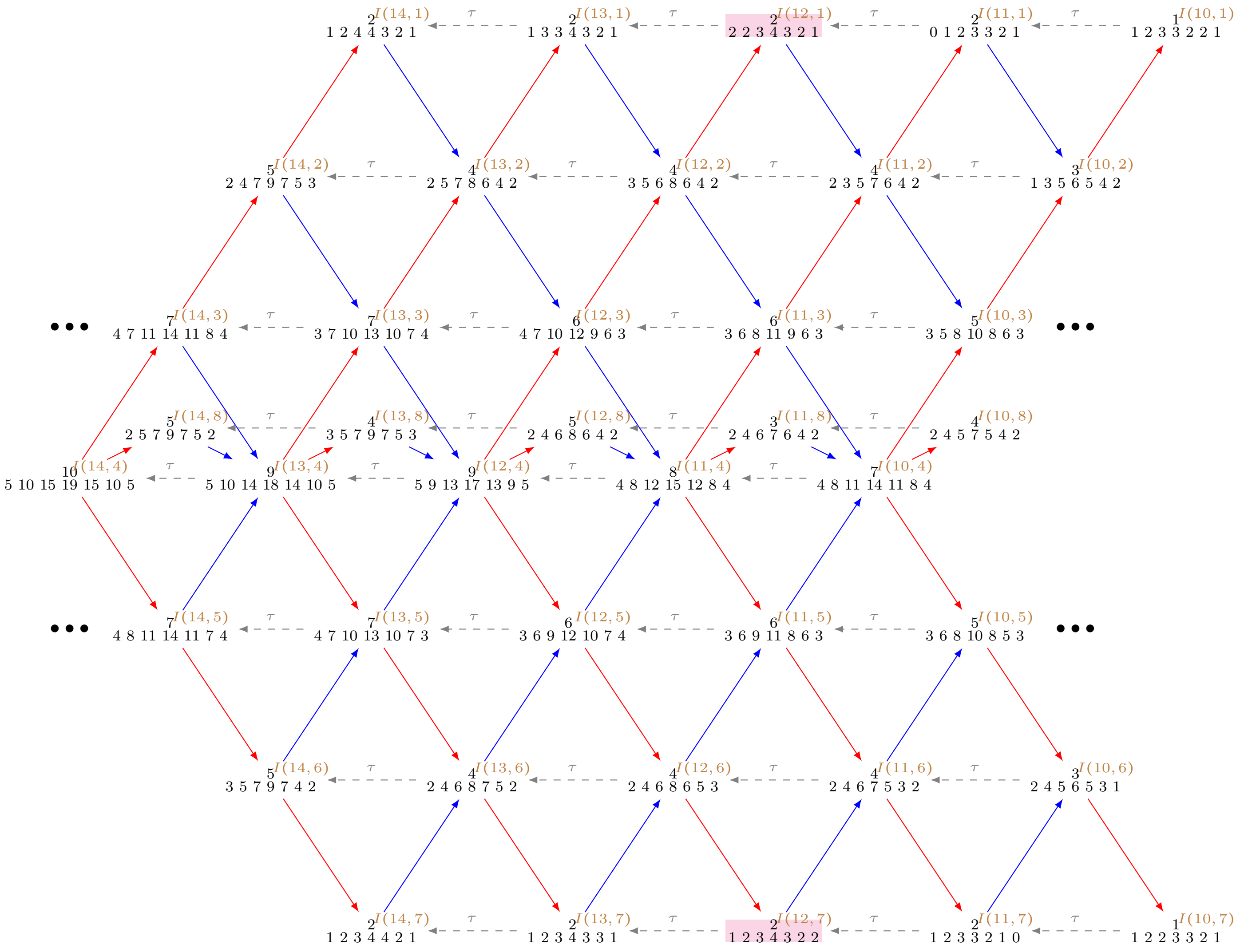
I(1,5)
I(0,5)
I(4,6)
I(3,6)
I(2,6)
I(1,6)
I(0,6)
I(4,7)
I(3,7)
I(2,7)
I(1,7)
I(0,7)
τ



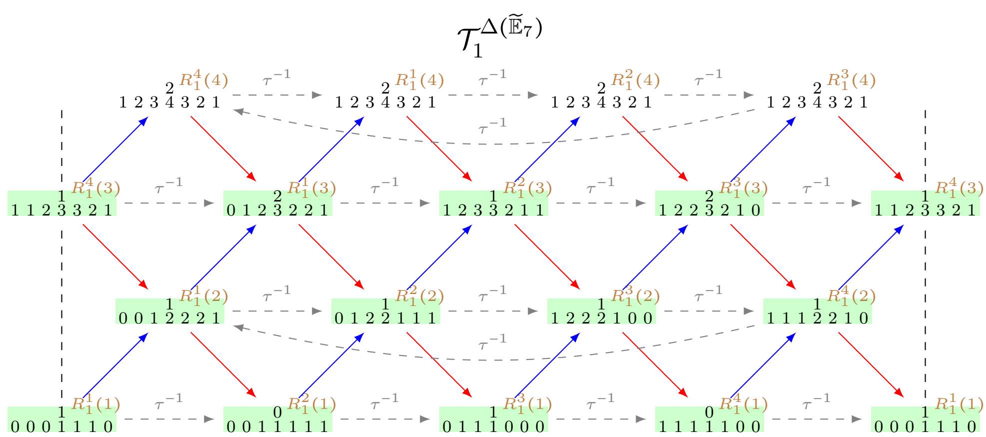


I(14,1) 2 1 2 4 4 3 2 1
I(13,1) 2 1 3 3 4 3 2 1
I(12,1) 2 2 2 3 4 3 2 1
I(11,1) 2 0 1 2 3 3 2 1
I(10,1) 1 1 2 3 3 2 2 1
I(14,2) 5 2 4 7 9 7 5 3
I(13,2) 4 2 5 7 8 6 4 2
I(12,2) 4 3 5 6 8 6 4 2
I(11,2) 4 2 3 5 7 6 4 2
I(10,2) 3 1 3 5 6 5 4 2
I(14,3) 7 4 7 11 14 11 8 4
I(13,3) 7 3 7 10 13 10 7 4
I(12,3) 6 4 7 10 12 9 6 3
I(11,3) 6 3 6 8 11 9 6 3
I(10,3) 5 3 5 8 10 8 6 3
I(14,8) 5 2 5 7 9 7 5 2
I(13,8) 4 3 5 7 9 7 5 3
I(12,8) 5 2 4 6 8 6 4 2
I(11,8) 3 2 4 6 7 6 4 2
I(10,8) 4 2 4 5 7 5 4 2
I(14,4) 10 5 10 15 19 15 10 5
I(13,4) 9 5 10 14 18 14 10 5
I(12,4) 9 5 9 13 17 13 9 5
I(11,4) 8 4 8 12 15 12 8 4
I(10,4) 7 4 8 11 14 11 8 4
I(14,5) 7 4 8 11 14 11 7 4
I(13,5) 7 4 7 10 13 10 7 3
I(12,5) 6 3 6 9 12 10 7 4
I(11,5) 6 3 6 9 11 8 6 3
I(10,5) 5 3 6 8 10 8 5 3
I(14,6) 5 3 5 7 9 7 4 2
I(13,6) 4 2 4 6 8 7 5 2
I(12,6) 4 2 4 6 8 6 5 3
I(11,6) 4 2 4 6 7 5 3 2
I(10,6) 3 2 4 5 6 5 3 1
I(14,7) 2 1 2 3 4 4 2 1
I(13,7) 2 1 2 3 4 3 3 1
I(12,7) 2 1 2 3 4 3 2 2
I(11,7) 2 1 2 3 3 2 1 0
I(10,7) 1 1 2 2 3 3 2 1
τ

Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{E}}_7)}$

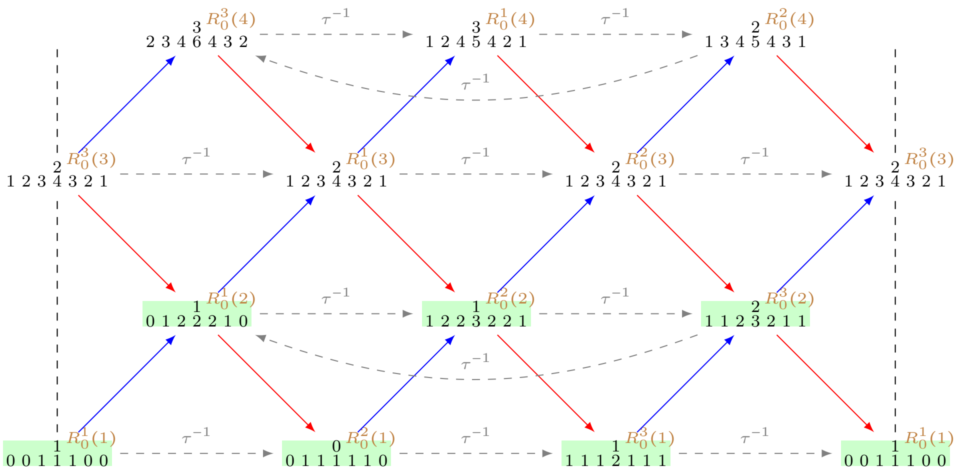
$R_1^1(1) : \big(I(0,8), P(0,6)\big),\ \big(I(1,6), P(0,8)\big),\ \big(I(1,7), P(1,1)\big)$
$R_1^1(2) : \big(R_1^2(1), R_1^1(1)\big),\ \big(I(5,7), P(0,7)\big),\ \big(I(1,6), P(1,6)\big),\ \big(I(3,1), P(2,1)\big),\ \big(I(1,7), P(4,7)\big)$
$R_1^2(1) : \big(I(0,5), P(0,3)\big),\ \big(I(2,1), P(0,7)\big),\ \big(I(0,6), P(1,8)\big),\ \big(I(0,7), P(2,1)\big)$
$R_1^2(2) : \big(R_1^3(1), R_1^2(1)\big),\ \big(I(2,2), P(0,2)\big),\ \big(I(4,7), P(1,7)\big),\ \big(I(0,6), P(2,6)\big),\ \big(I(2,1), P(3,1)\big)$
$\big(I(0,7), P(5,7)\big)$
$R_1^3(1) : \big(I(0,8), P(0,2)\big),\ \big(I(1,2), P(0,8)\big),\ \big(I(1,1), P(1,7)\big)$
$R_1^3(2) : \big(R_1^4(1), R_1^3(1)\big),\ \big(I(5,1), P(0,1)\big),\ \big(I(1,2), P(1,2)\big),\ \big(I(3,7), P(2,7)\big),\ \big(I(1,1), P(4,1)\big)$
$R_1^4(1) : \big(I(2,7), P(0,1)\big),\ \big(I(0,3), P(0,5)\big),\ \big(I(0,2), P(1,8)\big),\ \big(I(0,1), P(2,7)\big)$
$R_1^4(2) : \big(R_1^1(1), R_1^4(1)\big),\ \big(I(2,6), P(0,6)\big),\ \big(I(4,1), P(1,1)\big),\ \big(I(0,2), P(2,2)\big),\ \big(I(2,7), P(3,7)\big)$
$\big(I(0,1), P(5,1)\big)$
$R_1^4(3) : \big(R_1^1(2), R_1^4(1)\big),\ \big(R_1^2(1), R_1^4(2)\big),\ \big(I(8,1), P(0,7)\big),\ \big(I(6,7), P(2,1)\big),\ \big(I(4,1), P(4,7)\big)$
$\big(I(2,7), P(6,1)\big),\ \big(I(0,1), P(8,7)\big)$
$R_1^1(3) : \big(R_1^2(2), R_1^1(1)\big),\ \big(R_1^3(1), R_1^1(2)\big),\ \big(I(7,1), P(1,7)\big),\ \big(I(5,7), P(3,1)\big),\ \big(I(3,1), P(5,7)\big)$
$\big(I(1,7), P(7,1)\big)$
$R_1^2(3) : \big(R_1^3(2), R_1^2(1)\big),\ \big(R_1^4(1), R_1^2(2)\big),\ \big(I(8,7), P(0,1)\big),\ \big(I(6,1), P(2,7)\big),\ \big(I(4,7), P(4,1)\big)$
$\big(I(2,1), P(6,7)\big),\ \big(I(0,7), P(8,1)\big)$
$R_1^3(3) : \big(R_1^4(2), R_1^3(1)\big),\ \big(R_1^1(1), R_1^3(2)\big),\ \big(I(7,7), P(1,1)\big),\ \big(I(5,1), P(3,7)\big),\ \big(I(3,7), P(5,1)\big)$
$\big(I(1,1), P(7,7)\big)$



The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{E}}_7)}$

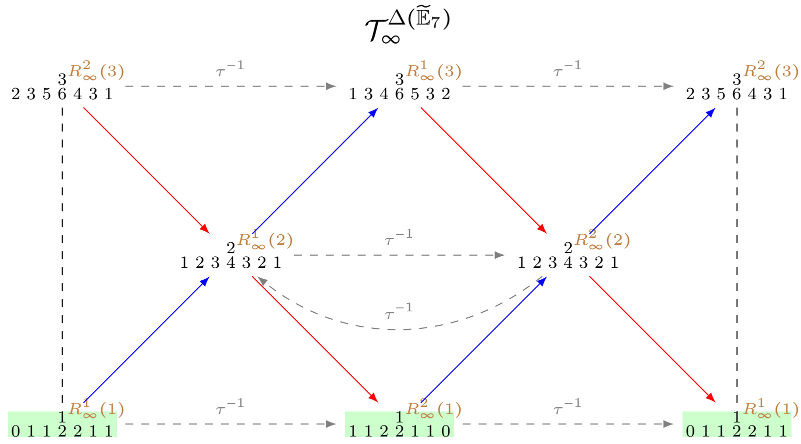
$R_0^1(1): \big(I(2,1), P(1,1)\big),\ \big(I(2,7), P(1,7)\big),\ \big(I(0,8), P(1,8)\big)$
$R_0^1(2): \big(R_0^2(1), R_0^1(1)\big),\ \big(I(5,1), P(2,1)\big),\ \big(I(5,7), P(2,7)\big),\ \big(I(2,1), P(5,1)\big),\ \big(I(2,7), P(5,7)\big)$
$R_0^2(1): \big(I(1,6), P(0,2)\big),\ \big(I(1,2), P(0,6)\big),\ \big(I(1,1), P(2,1)\big),\ \big(I(1,7), P(2,7)\big)$
$R_0^2(2): \big(R_0^3(1), R_0^2(1)\big),\ \big(I(7,1), P(0,1)\big),\ \big(I(7,7), P(0,7)\big),\ \big(I(4,1), P(3,1)\big),\ \big(I(4,7), P(3,7)\big)$
$\big(I(1,1), P(6,1)\big),\ \big(I(1,7), P(6,7)\big)$
$R_0^3(1): \big(I(3,1), P(0,1)\big),\ \big(I(3,7), P(0,7)\big),\ \big(I(1,8), P(0,8)\big),\ \big(I(0,6), P(1,2)\big),\ \big(I(0,2), P(1,6)\big)$
$\big(I(0,1), P(3,1)\big),\ \big(I(0,7), P(3,7)\big)$
$R_0^3(2): \big(R_0^1(1), R_0^3(1)\big),\ \big(I(6,1), P(1,1)\big),\ \big(I(6,7), P(1,7)\big),\ \big(I(3,1), P(4,1)\big),\ \big(I(3,7), P(4,7)\big)$
$\big(I(0,1), P(7,1)\big),\ \big(I(0,7), P(7,7)\big)$

$\mathcal{T}_0^{\Delta(\widetilde{\mathbb{E}}_7)}$

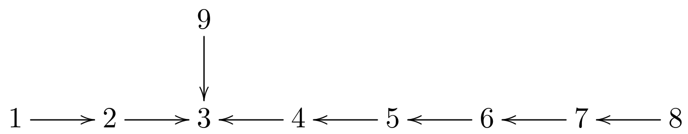


The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{E}}_7)}$

$R_\infty^1(1): \big(I(5,1), P(0,7)\big),\ \big(I(4,7), P(1,1)\big),\ \big(I(3,1), P(2,7)\big),\ \big(I(2,7), P(3,1)\big),\ \big(I(1,1), P(4,7)\big)$
$\big(I(0,7), P(5,1)\big)$
$R_\infty^2(1): \big(I(5,7), P(0,1)\big),\ \big(I(4,1), P(1,7)\big),\ \big(I(3,7), P(2,1)\big),\ \big(I(2,1), P(3,7)\big),\ \big(I(1,7), P(4,1)\big)$
$\big(I(0,1), P(5,7)\big)$



A.16 Schofield pairs for the quiver $\Delta(\widetilde{\mathbb{E}}_8)$ – $\delta = {}_{2\,4\,\overset{3}{6}\,5\,4\,3\,2\,1}$



$$C_{\Delta(\widetilde{\mathbb{E}}_8)} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1\\ 0&0&0&1&1&1&1&1&0\\ 0&0&0&0&1&1&1&1&0\\ 0&0&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \qquad \Phi_{\Delta(\widetilde{\mathbb{E}}_8)} = \begin{bmatrix} 0&0&-1&1&0&0&0&0&1\\ 1&0&-1&1&0&0&0&0&1\\ 0&1&-1&1&0&0&0&0&1\\ 0&1&-1&0&1&0&0&0&1\\ 0&1&-1&0&0&1&0&0&1\\ 0&1&-1&0&0&0&1&0&1\\ 0&1&-1&0&0&0&0&1&1\\ 0&1&-1&0&0&0&0&0&1\\ 0&1&-1&1&0&0&0&0&0 \end{bmatrix}$$

Schofield pairs associated to preprojective exceptional modules

Modules of the form $P(n,1)$

Defect: $\partial P(n,1) = -2$, for $n \geq 0$.

$P(0,1) : \big(I(0,1), P(0,2)\big),\ \big(I(0,2), P(0,3)\big)$
$P(1,1) : \big(I(0,9), P(0,4)\big),\ \big(I(4,8), P(0,9)\big)$
$P(2,1) : \big(I(3,7), P(0,2)\big),\ \big(I(1,1), P(0,5)\big),\ \big(I(3,8), P(1,9)\big)$
$P(3,1) : \big(R_1^1(1), P(0,1)\big),\ \big(I(5,8), P(0,6)\big),\ \big(I(8,8), P(0,9)\big),\ \big(I(2,7), P(1,2)\big),\ \big(I(0,1), P(1,5)\big)$
$\big(I(2,8), P(2,9)\big)$
$P(4,1) : \big(R_1^4(1), P(0,7)\big),\ \big(R_1^2(1), P(1,1)\big),\ \big(I(4,8), P(1,6)\big),\ \big(I(7,8), P(1,9)\big),\ \big(I(1,7), P(2,2)\big)$

$\big(I(1,8), P(3,9)\big)$

$P(5,1): \big(R_0^1(1), P(0,1)\big),\ \big(P(7,8), P(0,8)\big),\ \big(R_1^5(1), P(1,7)\big),\ \big(R_1^3(1), P(2,1)\big),\ \big(I(3,8), P(2,6)\big)$
$\big(I(6,8), P(2,9)\big),\ \big(I(0,7), P(3,2)\big),\ \big(I(0,8), P(4,9)\big)$

$P(6,1): \big(R_0^2(1), P(1,1)\big),\ \big(P(8,8), P(1,8)\big),\ \big(R_1^1(1), P(2,7)\big),\ \big(R_1^4(1), P(3,1)\big),\ \big(I(2,8), P(3,6)\big)$
$\big(I(5,8), P(3,9)\big)$

$P(7,1): \big(R_1^4(2), P(0,7)\big),\ \big(R_0^3(1), P(2,1)\big),\ \big(P(9,8), P(2,8)\big),\ \big(R_1^2(1), P(3,7)\big),\ \big(R_1^5(1), P(4,1)\big)$
$\big(I(1,8), P(4,6)\big),\ \big(I(4,8), P(4,9)\big)$

$P(8,1): \big(P(13,8), P(0,8)\big),\ \big(R_1^5(2), P(1,7)\big),\ \big(R_0^1(1), P(3,1)\big),\ \big(P(10,8), P(3,8)\big),\ \big(R_1^3(1), P(4,7)\big)$
$\big(R_1^1(1), P(5,1)\big),\ \big(I(0,8), P(5,6)\big),\ \big(I(3,8), P(5,9)\big)$

$P(9,1): \big(P(14,8), P(1,8)\big),\ \big(R_1^1(2), P(2,7)\big),\ \big(R_0^2(1), P(4,1)\big),\ \big(P(11,8), P(4,8)\big),\ \big(R_1^4(1), P(5,7)\big)$
$\big(R_1^2(1), P(6,1)\big),\ \big(I(2,8), P(6,9)\big)$

$P(10,1): \big(P(17,8), P(0,8)\big),\ \big(P(15,8), P(2,8)\big),\ \big(R_1^2(2), P(3,7)\big),\ \big(R_0^3(1), P(5,1)\big),\ \big(P(12,8), P(5,8)\big)$
$\big(R_1^5(1), P(6,7)\big),\ \big(R_1^3(1), P(7,1)\big),\ \big(I(1,8), P(7,9)\big)$

$P(11,1): \big(P(18,8), P(1,8)\big),\ \big(P(16,8), P(3,8)\big),\ \big(R_1^3(2), P(4,7)\big),\ \big(R_0^1(1), P(6,1)\big),\ \big(P(13,8), P(6,8)\big)$
$\big(R_1^1(1), P(7,7)\big),\ \big(R_1^4(1), P(8,1)\big),\ \big(I(0,8), P(8,9)\big)$

$P(12,1): \big(P(19,8), P(2,8)\big),\ \big(P(17,8), P(4,8)\big),\ \big(R_1^4(2), P(5,7)\big),\ \big(R_0^2(1), P(7,1)\big),\ \big(P(14,8), P(7,8)\big)$
$\big(R_1^2(1), P(8,7)\big),\ \big(R_1^5(1), P(9,1)\big)$

$P(13,1): \big(P(23,8), P(0,8)\big),\ \big(P(20,8), P(3,8)\big),\ \big(P(18,8), P(5,8)\big),\ \big(R_1^5(2), P(6,7)\big),\ \big(R_0^3(1), P(8,1)\big)$
$\big(P(15,8), P(8,8)\big),\ \big(R_1^3(1), P(9,7)\big),\ \big(R_1^1(1), P(10,1)\big)$

$P(n,1): \big(P(n+10,8), P(n-13,8)\big),\ \big(P(n+7,8), P(n-10,8)\big),\ \big(P(n+5,8), P(n-8,8)\big)$
$\big(R_1^{(n-9) \bmod 5+1}(2), P(n-7,7)\big),\ \big(R_0^{(n-11) \bmod 3+1}(1), P(n-5,1)\big),\ \big(P(n+2,8), P(n-5,8)\big)$
$\big(R_1^{(n-11) \bmod 5+1}(1), P(n-4,7)\big),\ \big(R_1^{(n-13) \bmod 5+1}(1), P(n-3,1)\big),\ n > 13$

Modules of the form $P(n,2)$

Defect: $\partial P(n,2) = -4$, for $n \geq 0$.

$P(0,2): \big(I(1,1), P(0,3)\big)$

$P(1,2): \big(P(1,1), P(0,1)\big),\ \big(I(5,8), P(0,4)\big),\ \big(P(2,8), P(0,9)\big),\ \big(I(0,1), P(1,3)\big)$

$P(2,2): \big(R_1^4(1), P(0,5)\big),\ \big(P(2,1), P(1,1)\big),\ \big(I(4,8), P(1,4)\big),\ \big(P(3,8), P(1,9)\big)$

$P(3,2): \big(P(4,7), P(0,1)\big),\ \big(P(7,8), P(0,6)\big),\ \big(R_1^5(1), P(1,5)\big),\ \big(P(3,1), P(2,1)\big),\ \big(I(3,8), P(2,4)\big)$
$\big(P(4,8), P(2,9)\big)$

$P(4,2): \big(P(6,1), P(0,7)\big),\ \big(P(5,7), P(1,1)\big),\ \big(P(8,8), P(1,6)\big),\ \big(R_1^1(1), P(2,5)\big),\ \big(P(4,1), P(3,1)\big)$
$\big(I(2,8), P(3,4)\big),\ \big(P(5,8), P(3,9)\big)$

$P(5,2): \big(P(6,9), P(0,8)\big),\ \big(P(7,1), P(1,7)\big),\ \big(P(6,7), P(2,1)\big),\ \big(P(9,8), P(2,6)\big),\ \big(R_1^2(1), P(3,5)\big)$
$\big(P(5,1), P(4,1)\big),\ \big(I(1,8), P(4,4)\big),\ \big(P(6,8), P(4,9)\big)$

$P(6,2):\big(P(7,9),P(1,8)\big),\ \big(P(8,1),P(2,7)\big),\ \big(P(7,7),P(3,1)\big),\ \big(P(10,8),P(3,6)\big),\ \big(R_1^3(1),P(4,5)\big)$
$\big(P(6,1),P(5,1)\big),\ \big(I(0,8),P(5,4)\big),\ \big(P(7,8),P(5,9)\big)$

$P(7,2):\big(P(8,9),P(2,8)\big),\ \big(P(9,1),P(3,7)\big),\ \big(P(8,7),P(4,1)\big),\ \big(P(11,8),P(4,6)\big),\ \big(R_1^4(1),P(5,5)\big)$
$\big(P(7,1),P(6,1)\big),\ \big(P(8,8),P(6,9)\big)$

$P(8,2):\big(P(9,6),P(0,8)\big),\ \big(P(9,9),P(3,8)\big),\ \big(P(10,1),P(4,7)\big),\ \big(P(9,7),P(5,1)\big),\ \big(P(12,8),P(5,6)\big)$
$\big(R_1^5(1),P(6,5)\big),\ \big(P(8,1),P(7,1)\big),\ \big(P(9,8),P(7,9)\big)$

$P(n,2):\big(P(n+1,6),P(n-8,8)\big),\ \big(P(n+1,9),P(n-5,8)\big),\ \big(P(n+2,1),P(n-4,7)\big)$
$\big(P(n+1,7),P(n-3,1)\big),\ \big(P(n+4,8),P(n-3,6)\big),\ \big(R_1^{(n-4)\text{ mod }5+1}(1),P(n-2,5)\big)$
$\big(P(n,1),P(n-1,1)\big),\ \big(P(n+1,8),P(n-1,9)\big),\ n>8$

Modules of the form $P(n,3)$

Defect: $\partial P(n,3)=-6$, for $n\geq 0$.

$P(0,3):-$

$P(1,3):\big(P(1,1),P(0,2)\big),\ \big(P(1,8),P(0,4)\big),\ \big(P(1,9),P(0,9)\big)$

$P(2,3):\big(P(2,2),P(0,1)\big),\ \big(P(2,7),P(0,5)\big),\ \big(P(2,1),P(1,2)\big),\ \big(P(2,8),P(1,4)\big),\ \big(P(2,9),P(1,9)\big)$

$P(3,3):\big(P(3,6),P(0,6)\big),\ \big(P(3,2),P(1,1)\big),\ \big(P(3,7),P(1,5)\big),\ \big(P(3,1),P(2,2)\big),\ \big(P(3,8),P(2,4)\big)$
$\big(P(3,9),P(2,9)\big)$

$P(4,3):\big(P(4,5),P(0,7)\big),\ \big(P(4,6),P(1,6)\big),\ \big(P(4,2),P(2,1)\big),\ \big(P(4,7),P(2,5)\big),\ \big(P(4,1),P(3,2)\big)$
$\big(P(4,8),P(3,4)\big),\ \big(P(4,9),P(3,9)\big)$

$P(5,3):\big(P(5,4),P(0,8)\big),\ \big(P(5,5),P(1,7)\big),\ \big(P(5,6),P(2,6)\big),\ \big(P(5,2),P(3,1)\big),\ \big(P(5,7),P(3,5)\big)$
$\big(P(5,1),P(4,2)\big),\ \big(P(5,8),P(4,4)\big),\ \big(P(5,9),P(4,9)\big)$

$P(n,3):\big(P(n,4),P(n-5,8)\big),\ \big(P(n,5),P(n-4,7)\big),\ \big(P(n,6),P(n-3,6)\big)$
$\big(P(n,2),P(n-2,1)\big),\ \big(P(n,7),P(n-2,5)\big),\ \big(P(n,1),P(n-1,2)\big)$
$\big(P(n,8),P(n-1,4)\big),\ \big(P(n,9),P(n-1,9)\big),\ n>5$

Modules of the form $P(n,4)$

Defect: $\partial P(n,4)=-5$, for $n\geq 0$.

$P(0,4):\big(I(4,8),P(0,3)\big)$

$P(1,4):\big(P(3,8),P(0,2)\big),\ \big(P(1,8),P(0,5)\big),\ \big(P(2,1),P(0,9)\big),\ \big(I(3,8),P(1,3)\big)$

$P(2,4):\big(P(3,9),P(0,1)\big),\ \big(P(2,7),P(0,6)\big),\ \big(P(4,8),P(1,2)\big),\ \big(P(2,8),P(1,5)\big),\ \big(P(3,1),P(1,9)\big)$
$\big(I(2,8),P(2,3)\big)$

$P(3,4):\big(P(3,6),P(0,7)\big),\ \big(P(4,9),P(1,1)\big),\ \big(P(3,7),P(1,6)\big),\ \big(P(5,8),P(2,2)\big),\ \big(P(3,8),P(2,5)\big)$

$\big(P(4,1), P(2,9)\big),\ \big(I(1,8), P(3,3)\big)$
$P(4,4) : \big(P(4,5), P(0,8)\big),\ \big(P(4,6), P(1,7)\big),\ \big(P(5,9), P(2,1)\big),\ \big(P(4,7), P(2,6)\big),\ \big(P(6,8), P(3,2)\big)$
$\big(P(4,8), P(3,5)\big),\ \big(P(5,1), P(3,9)\big),\ \big(I(0,8), P(4,3)\big)$
$P(5,4) : \big(P(5,5), P(1,8)\big),\ \big(P(5,6), P(2,7)\big),\ \big(P(6,9), P(3,1)\big),\ \big(P(5,7), P(3,6)\big),\ \big(P(7,8), P(4,2)\big)$
$\big(P(5,8), P(4,5)\big),\ \big(P(6,1), P(4,9)\big)$
$P(6,4) : \big(P(7,2), P(0,8)\big),\ \big(P(6,5), P(2,8)\big),\ \big(P(6,6), P(3,7)\big),\ \big(P(7,9), P(4,1)\big),\ \big(P(6,7), P(4,6)\big)$
$\big(P(8,8), P(5,2)\big),\ \big(P(6,8), P(5,5)\big),\ \big(P(7,1), P(5,9)\big)$
$P(n,4) : \big(P(n+1,2), P(n-6,8)\big),\ \big(P(n,5), P(n-4,8)\big),\ \big(P(n,6), P(n-3,7)\big)$
$\big(P(n+1,9), P(n-2,1)\big),\ \big(P(n,7), P(n-2,6)\big),\ \big(P(n+2,8), P(n-1,2)\big)$
$\big(P(n,8), P(n-1,5)\big),\ \big(P(n+1,1), P(n-1,9)\big),\ n > 6$

Modules of the form $P(n,5)$

Defect: $\partial P(n,5) = -4$, for $n \geq 0$.

$P(0,5) : \big(I(3,7), P(0,3)\big),\ \big(I(3,8), P(0,4)\big)$
$P(1,5) : \big(R_1^1(1), P(0,2)\big),\ \big(P(1,8), P(0,6)\big),\ \big(P(4,8), P(0,9)\big),\ \big(I(2,7), P(1,3)\big),\ \big(I(2,8), P(1,4)\big)$
$P(2,5) : \big(P(4,1), P(0,1)\big),\ \big(P(2,7), P(0,7)\big),\ \big(R_1^2(1), P(1,2)\big),\ \big(P(2,8), P(1,6)\big),\ \big(P(5,8), P(1,9)\big)$
$\big(I(1,7), P(2,3)\big),\ \big(I(1,8), P(2,4)\big)$
$P(3,5) : \big(P(3,6), P(0,8)\big),\ \big(P(5,1), P(1,1)\big),\ \big(P(3,7), P(1,7)\big),\ \big(R_1^3(1), P(2,2)\big),\ \big(P(3,8), P(2,6)\big)$
$\big(P(6,8), P(2,9)\big),\ \big(I(0,7), P(3,3)\big),\ \big(I(0,8), P(3,4)\big)$
$P(4,5) : \big(P(4,6), P(1,8)\big),\ \big(P(6,1), P(2,1)\big),\ \big(P(4,7), P(2,7)\big),\ \big(R_1^4(1), P(3,2)\big),\ \big(P(4,8), P(3,6)\big)$
$\big(P(7,8), P(3,9)\big)$
$P(5,5) : \big(P(8,7), P(0,7)\big),\ \big(P(5,6), P(2,8)\big),\ \big(P(7,1), P(3,1)\big),\ \big(P(5,7), P(3,7)\big),\ \big(R_1^5(1), P(4,2)\big)$
$\big(P(5,8), P(4,6)\big),\ \big(P(8,8), P(4,9)\big)$
$P(6,5) : \big(P(8,9), P(0,8)\big),\ \big(P(9,7), P(1,7)\big),\ \big(P(6,6), P(3,8)\big),\ \big(P(8,1), P(4,1)\big),\ \big(P(6,7), P(4,7)\big)$
$\big(R_1^1(1), P(5,2)\big),\ \big(P(6,8), P(5,6)\big),\ \big(P(9,8), P(5,9)\big)$
$P(n,5) : \big(P(n+2,9), P(n-6,8)\big),\ \big(P(n+3,7), P(n-5,7)\big),\ \big(P(n,6), P(n-3,8)\big)$
$\big(P(n+2,1), P(n-2,1)\big),\ \big(P(n,7), P(n-2,7)\big),\ \big(R_1^{(n-6) \bmod 5+1}(1), P(n-1,2)\big)$
$\big(P(n,8), P(n-1,6)\big),\ \big(P(n+3,8), P(n-1,9)\big),\ n > 6$

Modules of the form $P(n,6)$

Defect: $\partial P(n,6) = -3$, for $n \geq 0$.

$P(0,6) : \big(I(2,6), P(0,3)\big),\ \big(I(2,7), P(0,4)\big),\ \big(I(2,8), P(0,5)\big)$

$P(1,6):\big(I(7,8),P(0,2)\big),\ \big(P(1,8),P(0,7)\big),\ \big(R_1^2(1),P(0,9)\big),\ \big(I(1,6),P(1,3)\big),\ \big(I(1,7),P(1,4)\big)$
$\big(I(1,8),P(1,5)\big)$

$P(2,6):\big(P(6,8),P(0,1)\big),\ \big(P(2,7),P(0,8)\big),\ \big(I(6,8),P(1,2)\big),\ \big(P(2,8),P(1,7)\big),\ \big(R_1^3(1),P(1,9)\big)$
$\big(I(0,6),P(2,3)\big),\ \big(I(0,7),P(2,4)\big),\ \big(I(0,8),P(2,5)\big)$

$P(3,6):\big(P(7,8),P(1,1)\big),\ \big(P(3,7),P(1,8)\big),\ \big(I(5,8),P(2,2)\big),\ \big(P(3,8),P(2,7)\big),\ \big(R_1^4(1),P(2,9)\big)$

$P(4,6):\big(R_1^4(2),P(0,6)\big),\ \big(P(8,8),P(2,1)\big),\ \big(P(4,7),P(2,8)\big),\ \big(I(4,8),P(3,2)\big),\ \big(P(4,8),P(3,7)\big)$
$\big(R_1^5(1),P(3,9)\big)$

$P(5,6):\big(P(13,8),P(0,7)\big),\ \big(R_1^5(2),P(1,6)\big),\ \big(P(9,8),P(3,1)\big),\ \big(P(5,7),P(3,8)\big),\ \big(I(3,8),P(4,2)\big)$
$\big(P(5,8),P(4,7)\big),\ \big(R_1^1(1),P(4,9)\big)$

$P(6,6):\big(P(9,1),P(0,8)\big),\ \big(P(14,8),P(1,7)\big),\ \big(R_1^1(2),P(2,6)\big),\ \big(P(10,8),P(4,1)\big),\ \big(P(6,7),P(4,8)\big)$
$\big(I(2,8),P(5,2)\big),\ \big(P(6,8),P(5,7)\big),\ \big(R_1^2(1),P(5,9)\big)$

$P(7,6):\big(P(10,1),P(1,8)\big),\ \big(P(15,8),P(2,7)\big),\ \big(R_1^2(2),P(3,6)\big),\ \big(P(11,8),P(5,1)\big),\ \big(P(7,7),P(5,8)\big)$
$\big(I(1,8),P(6,2)\big),\ \big(P(7,8),P(6,7)\big),\ \big(R_1^3(1),P(6,9)\big)$

$P(8,6):\big(P(11,1),P(2,8)\big),\ \big(P(16,8),P(3,7)\big),\ \big(R_1^3(2),P(4,6)\big),\ \big(P(12,8),P(6,1)\big),\ \big(P(8,7),P(6,8)\big)$
$\big(I(0,8),P(7,2)\big),\ \big(P(8,8),P(7,7)\big),\ \big(R_1^4(1),P(7,9)\big)$

$P(9,6):\big(P(12,1),P(3,8)\big),\ \big(P(17,8),P(4,7)\big),\ \big(R_1^4(2),P(5,6)\big),\ \big(P(13,8),P(7,1)\big),\ \big(P(9,7),P(7,8)\big)$
$\big(P(9,8),P(8,7)\big),\ \big(R_1^5(1),P(8,9)\big)$

$P(10,6):\big(P(14,7),P(0,8)\big),\ \big(P(13,1),P(4,8)\big),\ \big(P(18,8),P(5,7)\big),\ \big(R_1^5(2),P(6,6)\big),\ \big(P(14,8),P(8,1)\big)$
$\big(P(10,7),P(8,8)\big),\ \big(P(10,8),P(9,7)\big),\ \big(R_1^1(1),P(9,9)\big)$

$P(n,6):\big(P(n+4,7),P(n-10,8)\big),\ \big(P(n+3,1),P(n-6,8)\big),\ \big(P(n+8,8),P(n-5,7)\big)$
$\big(R_1^{(n-6) \bmod 5+1}(2),P(n-4,6)\big),\ \big(P(n+4,8),P(n-2,1)\big),\ \big(P(n,7),P(n-2,8)\big)$
$\big(P(n,8),P(n-1,7)\big),\ \big(R_1^{(n-10) \bmod 5+1}(1),P(n-1,9)\big),\ n>10$

Modules of the form $P(n,7)$

Defect: $\partial P(n,7)=-2$, for $n\geq 0$.

$P(0,7):\big(I(1,5),P(0,3)\big),\ \big(I(1,6),P(0,4)\big),\ \big(I(1,7),P(0,5)\big),\ \big(I(1,8),P(0,6)\big)$

$P(1,7):\big(I(2,1),P(0,2)\big),\ \big(P(1,8),P(0,8)\big),\ \big(I(6,8),P(0,9)\big),\ \big(I(0,5),P(1,3)\big),\ \big(I(0,6),P(1,4)\big)$
$\big(I(0,7),P(1,5)\big),\ \big(I(0,8),P(1,6)\big)$

$P(2,7):\big(R_1^4(1),P(0,1)\big),\ \big(I(1,1),P(1,2)\big),\ \big(P(2,8),P(1,8)\big),\ \big(I(5,8),P(1,9)\big)$

$P(3,7):\big(I(4,7),P(0,5)\big),\ \big(R_1^5(1),P(1,1)\big),\ \big(I(0,1),P(2,2)\big),\ \big(P(3,8),P(2,8)\big),\ \big(I(4,8),P(2,9)\big)$

$P(4,7):\big(I(9,8),P(0,6)\big),\ \big(I(3,7),P(1,5)\big),\ \big(R_1^1(1),P(2,1)\big),\ \big(P(4,8),P(3,8)\big),\ \big(I(3,8),P(3,9)\big)$

$P(5,7):\big(R_1^1(2),P(0,1)\big),\ \big(R_0^2(1),P(0,7)\big),\ \big(I(8,8),P(1,6)\big),\ \big(I(2,7),P(2,5)\big),\ \big(R_1^2(1),P(3,1)\big)$
$\big(P(5,8),P(4,8)\big),\ \big(I(2,8),P(4,9)\big)$

$P(6,7):\big(P(11,8),P(0,8)\big),\ \big(R_1^2(2),P(1,1)\big),\ \big(R_0^3(1),P(1,7)\big),\ \big(I(7,8),P(2,6)\big),\ \big(I(1,7),P(3,5)\big)$

$\big(R_1^3(1), P(4,1)\big),\ \big(P(6,8), P(5,8)\big),\ \big(I(1,8), P(5,9)\big)$

$P(7,7) : \big(P(12,8), P(1,8)\big),\ \big(R_1^3(2), P(2,1)\big),\ \big(R_0^1(1), P(2,7)\big),\ \big(I(6,8), P(3,6)\big),\ \big(I(0,7), P(4,5)\big)$
$\big(R_1^4(1), P(5,1)\big),\ \big(P(7,8), P(6,8)\big),\ \big(I(0,8), P(6,9)\big)$

$P(8,7) : \big(P(13,8), P(2,8)\big),\ \big(R_1^4(2), P(3,1)\big),\ \big(R_0^2(1), P(3,7)\big),\ \big(I(5,8), P(4,6)\big),\ \big(R_1^5(1), P(6,1)\big)$
$\big(P(8,8), P(7,8)\big)$

$P(9,7) : \big(R_1^4(3), P(0,7)\big),\ \big(P(14,8), P(3,8)\big),\ \big(R_1^5(2), P(4,1)\big),\ \big(R_0^3(1), P(4,7)\big),\ \big(I(4,8), P(5,6)\big)$
$\big(R_1^1(1), P(7,1)\big),\ \big(P(9,8), P(8,8)\big)$

$P(10,7) : \big(P(19,8), P(0,8)\big),\ \big(R_1^5(3), P(1,7)\big),\ \big(P(15,8), P(4,8)\big),\ \big(R_1^1(2), P(5,1)\big),\ \big(R_0^1(1), P(5,7)\big)$
$\big(I(3,8), P(6,6)\big),\ \big(R_1^2(1), P(8,1)\big),\ \big(P(10,8), P(9,8)\big)$

$P(11,7) : \big(P(20,8), P(1,8)\big),\ \big(R_1^1(3), P(2,7)\big),\ \big(P(16,8), P(5,8)\big),\ \big(R_1^2(2), P(6,1)\big),\ \big(R_0^2(1), P(6,7)\big)$
$\big(I(2,8), P(7,6)\big),\ \big(R_1^3(1), P(9,1)\big),\ \big(P(11,8), P(10,8)\big)$

$P(12,7) : \big(P(21,8), P(2,8)\big),\ \big(R_1^2(3), P(3,7)\big),\ \big(P(17,8), P(6,8)\big),\ \big(R_1^3(2), P(7,1)\big),\ \big(R_0^3(1), P(7,7)\big)$
$\big(I(1,8), P(8,6)\big),\ \big(R_1^4(1), P(10,1)\big),\ \big(P(12,8), P(11,8)\big)$

$P(13,7) : \big(P(22,8), P(3,8)\big),\ \big(R_1^3(3), P(4,7)\big),\ \big(P(18,8), P(7,8)\big),\ \big(R_1^4(2), P(8,1)\big),\ \big(R_0^1(1), P(8,7)\big)$
$\big(I(0,8), P(9,6)\big),\ \big(R_1^5(1), P(11,1)\big),\ \big(P(13,8), P(12,8)\big)$

$P(14,7) : \big(P(23,8), P(4,8)\big),\ \big(R_1^4(3), P(5,7)\big),\ \big(P(19,8), P(8,8)\big),\ \big(R_1^5(2), P(9,1)\big),\ \big(R_0^2(1), P(9,7)\big)$
$\big(R_1^1(1), P(12,1)\big),\ \big(P(14,8), P(13,8)\big)$

$P(15,7) : \big(P(29,8), P(0,8)\big),\ \big(P(24,8), P(5,8)\big),\ \big(R_1^5(3), P(6,7)\big),\ \big(P(20,8), P(9,8)\big),\ \big(R_1^1(2), P(10,1)\big)$
$\big(R_0^3(1), P(10,7)\big),\ \big(R_1^2(1), P(13,1)\big),\ \big(P(15,8), P(14,8)\big)$

$P(n,7) : \big(P(n+14,8), P(n-15,8)\big),\ \big(P(n+9,8), P(n-10,8)\big),\ \big(R_1^{(n-11) \bmod 5+1}(3), P(n-9,7)\big)$
$\big(P(n+5,8), P(n-6,8)\big),\ \big(R_1^{(n-15) \bmod 5+1}(2), P(n-5,1)\big),\ \big(R_0^{(n-13) \bmod 3+1}(1), P(n-5,7)\big)$
$\big(R_1^{(n-14) \bmod 5+1}(1), P(n-2,1)\big),\ \big(P(n,8), P(n-1,8)\big),\ n > 15$

Modules of the form $P(n,8)$

Defect: $\partial P(n,8) = -1$, for $n \geq 0$.

$P(0,8) : \big(I(0,4), P(0,3)\big),\ \big(I(0,5), P(0,4)\big),\ \big(I(0,6), P(0,5)\big),\ \big(I(0,7), P(0,6)\big),\ \big(I(0,8), P(0,7)\big)$

$P(1,8) : \big(I(0,9), P(0,2)\big),\ \big(I(1,1), P(0,9)\big)$

$P(2,8) : \big(I(4,8), P(0,1)\big),\ \big(I(0,2), P(0,4)\big),\ \big(I(0,1), P(1,9)\big)$

$P(3,8) : \big(I(0,9), P(0,5)\big),\ \big(I(3,7), P(0,9)\big),\ \big(I(3,8), P(1,1)\big)$

$P(4,8) : \big(I(2,6), P(0,2)\big),\ \big(I(1,1), P(0,6)\big),\ \big(I(2,7), P(1,9)\big),\ \big(I(2,8), P(2,1)\big)$

$P(5,8) : \big(I(7,8), P(0,1)\big),\ \big(I(5,8), P(0,7)\big),\ \big(I(3,1), P(0,9)\big),\ \big(I(1,6), P(1,2)\big),\ \big(I(0,1), P(1,6)\big)$
$\big(I(1,7), P(2,9)\big),\ \big(I(1,8), P(3,1)\big)$

$P(6,8) : \big(R_1^4(1), P(0,8)\big),\ \big(I(6,8), P(1,1)\big),\ \big(I(4,8), P(1,7)\big),\ \big(I(2,1), P(1,9)\big),\ \big(I(0,6), P(2,2)\big)$

$\big(I(0,7),P(3,9)\big),\ \big(I(0,8),P(4,1)\big)$

$P(7,8):\big(I(9,8),P(0,1)\big),\ \big(R_1^5(1),P(1,8)\big),\ \big(I(5,8),P(2,1)\big),\ \big(I(3,8),P(2,7)\big),\ \big(I(1,1),P(2,9)\big)$

$P(8,8):\big(I(4,7),P(0,6)\big),\ \big(I(8,8),P(1,1)\big),\ \big(R_1^1(1),P(2,8)\big),\ \big(I(4,8),P(3,1)\big),\ \big(I(2,8),P(3,7)\big)$
$\big(I(0,1),P(3,9)\big)$

$P(9,8):\big(I(9,8),P(0,7)\big),\ \big(I(3,7),P(1,6)\big),\ \big(I(7,8),P(2,1)\big),\ \big(R_1^2(1),P(3,8)\big),\ \big(I(3,8),P(4,1)\big)$
$\big(I(1,8),P(4,7)\big)$

$P(10,8):\big(I(12,8),P(0,1)\big),\ \big(R_0^2(1),P(0,8)\big),\ \big(I(8,8),P(1,7)\big),\ \big(I(2,7),P(2,6)\big),\ \big(I(6,8),P(3,1)\big)$
$\big(R_1^3(1),P(4,8)\big),\ \big(I(2,8),P(5,1)\big),\ \big(I(0,8),P(5,7)\big)$

$P(11,8):\big(I(11,8),P(1,1)\big),\ \big(R_0^3(1),P(1,8)\big),\ \big(I(7,8),P(2,7)\big),\ \big(I(1,7),P(3,6)\big),\ \big(I(5,8),P(4,1)\big)$
$\big(R_1^4(1),P(5,8)\big),\ \big(I(1,8),P(6,1)\big)$

$P(12,8):\big(R_1^4(2),P(0,8)\big),\ \big(I(10,8),P(2,1)\big),\ \big(R_0^1(1),P(2,8)\big),\ \big(I(6,8),P(3,7)\big),\ \big(I(0,7),P(4,6)\big)$
$\big(I(4,8),P(5,1)\big),\ \big(R_1^5(1),P(6,8)\big),\ \big(I(0,8),P(7,1)\big)$

$P(13,8):\big(R_1^5(2),P(1,8)\big),\ \big(I(9,8),P(3,1)\big),\ \big(R_0^2(1),P(3,8)\big),\ \big(I(5,8),P(4,7)\big),\ \big(I(3,8),P(6,1)\big)$
$\big(R_1^1(1),P(7,8)\big)$

$P(14,8):\big(I(14,8),P(0,7)\big),\ \big(R_1^1(2),P(2,8)\big),\ \big(I(8,8),P(4,1)\big),\ \big(R_0^3(1),P(4,8)\big),\ \big(I(4,8),P(5,7)\big)$
$\big(I(2,8),P(7,1)\big),\ \big(R_1^2(1),P(8,8)\big)$

$P(15,8):\big(R_\infty^1(1),P(0,8)\big),\ \big(I(13,8),P(1,7)\big),\ \big(R_1^2(2),P(3,8)\big),\ \big(I(7,8),P(5,1)\big),\ \big(R_0^1(1),P(5,8)\big)$
$\big(I(3,8),P(6,7)\big),\ \big(I(1,8),P(8,1)\big),\ \big(R_1^3(1),P(9,8)\big)$

$P(16,8):\big(R_\infty^2(1),P(1,8)\big),\ \big(I(12,8),P(2,7)\big),\ \big(R_1^3(2),P(4,8)\big),\ \big(I(6,8),P(6,1)\big),\ \big(R_0^2(1),P(6,8)\big)$
$\big(I(2,8),P(7,7)\big),\ \big(I(0,8),P(9,1)\big),\ \big(R_1^4(1),P(10,8)\big)$

$P(17,8):\big(R_\infty^1(1),P(2,8)\big),\ \big(I(11,8),P(3,7)\big),\ \big(R_1^4(2),P(5,8)\big),\ \big(I(5,8),P(7,1)\big),\ \big(R_0^3(1),P(7,8)\big)$
$\big(I(1,8),P(8,7)\big),\ \big(R_1^5(1),P(11,8)\big)$

$P(18,8):\big(R_1^4(3),P(0,8)\big),\ \big(R_\infty^2(1),P(3,8)\big),\ \big(I(10,8),P(4,7)\big),\ \big(R_1^5(2),P(6,8)\big),\ \big(I(4,8),P(8,1)\big)$
$\big(R_0^1(1),P(8,8)\big),\ \big(I(0,8),P(9,7)\big),\ \big(R_1^1(1),P(12,8)\big)$

$P(19,8):\big(R_1^5(3),P(1,8)\big),\ \big(R_\infty^1(1),P(4,8)\big),\ \big(I(9,8),P(5,7)\big),\ \big(R_1^1(2),P(7,8)\big),\ \big(I(3,8),P(9,1)\big)$
$\big(R_0^2(1),P(9,8)\big),\ \big(R_1^2(1),P(13,8)\big)$

$P(20,8):\big(R_0^2(2),P(0,8)\big),\ \big(R_1^1(3),P(2,8)\big),\ \big(R_\infty^2(1),P(5,8)\big),\ \big(I(8,8),P(6,7)\big),\ \big(R_1^2(2),P(8,8)\big)$
$\big(I(2,8),P(10,1)\big),\ \big(R_0^3(1),P(10,8)\big),\ \big(R_1^3(1),P(14,8)\big)$

$P(21,8):\big(R_0^3(2),P(1,8)\big),\ \big(R_1^2(3),P(3,8)\big),\ \big(R_\infty^1(1),P(6,8)\big),\ \big(I(7,8),P(7,7)\big),\ \big(R_1^3(2),P(9,8)\big)$
$\big(I(1,8),P(11,1)\big),\ \big(R_0^1(1),P(11,8)\big),\ \big(R_1^4(1),P(15,8)\big)$

$P(22,8):\big(R_0^1(2),P(2,8)\big),\ \big(R_1^3(3),P(4,8)\big),\ \big(R_\infty^2(1),P(7,8)\big),\ \big(I(6,8),P(8,7)\big),\ \big(R_1^4(2),P(10,8)\big)$
$\big(I(0,8),P(12,1)\big),\ \big(R_0^2(1),P(12,8)\big),\ \big(R_1^5(1),P(16,8)\big)$

$P(23,8):\big(R_0^2(2),P(3,8)\big),\ \big(R_1^4(3),P(5,8)\big),\ \big(R_\infty^1(1),P(8,8)\big),\ \big(I(5,8),P(9,7)\big),\ \big(R_1^5(2),P(11,8)\big)$
$\big(R_0^3(1),P(13,8)\big),\ \big(R_1^1(1),P(17,8)\big)$

$P(24,8):\big(R_1^4(4),P(0,8)\big),\ \big(R_0^3(2),P(4,8)\big),\ \big(R_1^5(3),P(6,8)\big),\ \big(R_\infty^2(1),P(9,8)\big),\ \big(I(4,8),P(10,7)\big)$

$\big(R_1^1(2), P(12,8)\big),\ \big(R_0^1(1), P(14,8)\big),\ \big(R_1^2(1), P(18,8)\big)$

$P(25,8) : \big(R_1^5(4), P(1,8)\big),\ \big(R_0^1(2), P(5,8)\big),\ \big(R_1^1(3), P(7,8)\big),\ \big(R_\infty^1(1), P(10,8)\big),\ \big(I(3,8), P(11,7)\big)$
$\big(R_1^2(2), P(13,8)\big),\ \big(R_0^2(1), P(15,8)\big),\ \big(R_1^3(1), P(19,8)\big)$

$P(26,8) : \big(R_1^1(4), P(2,8)\big),\ \big(R_0^2(2), P(6,8)\big),\ \big(R_1^2(3), P(8,8)\big),\ \big(R_\infty^2(1), P(11,8)\big),\ \big(I(2,8), P(12,7)\big)$
$\big(R_1^3(2), P(14,8)\big),\ \big(R_0^3(1), P(16,8)\big),\ \big(R_1^4(1), P(20,8)\big)$

$P(27,8) : \big(R_1^2(4), P(3,8)\big),\ \big(R_0^3(2), P(7,8)\big),\ \big(R_1^3(3), P(9,8)\big),\ \big(R_\infty^1(1), P(12,8)\big),\ \big(I(1,8), P(13,7)\big)$
$\big(R_1^4(2), P(15,8)\big),\ \big(R_0^1(1), P(17,8)\big),\ \big(R_1^5(1), P(21,8)\big)$

$P(28,8) : \big(R_1^3(4), P(4,8)\big),\ \big(R_0^1(2), P(8,8)\big),\ \big(R_1^4(3), P(10,8)\big),\ \big(R_\infty^2(1), P(13,8)\big),\ \big(I(0,8), P(14,7)\big)$
$\big(R_1^5(2), P(16,8)\big),\ \big(R_0^2(1), P(18,8)\big),\ \big(R_1^1(1), P(22,8)\big)$

$P(29,8) : \big(R_1^4(4), P(5,8)\big),\ \big(R_0^2(2), P(9,8)\big),\ \big(R_1^5(3), P(11,8)\big),\ \big(R_\infty^1(1), P(14,8)\big),\ \big(R_1^1(2), P(17,8)\big)$
$\big(R_0^3(1), P(19,8)\big),\ \big(R_1^2(1), P(23,8)\big)$

$P(30,8) : \big(R_1^5(4), P(6,8)\big),\ \big(R_0^3(2), P(10,8)\big),\ \big(R_1^1(3), P(12,8)\big),\ \big(R_\infty^2(1), P(15,8)\big),\ \big(R_1^2(2), P(18,8)\big)$
$\big(R_0^1(1), P(20,8)\big),\ \big(R_1^3(1), P(24,8)\big),\ \big(I(29,8), 2P(0,8)\big)$

$P(n,8) : \big(R_1^{(n-26) \bmod 5+1}(4), P(n-24,8)\big),\ \big(R_0^{(n-28) \bmod 3+1}(2), P(n-20,8)\big),\ \big(R_1^{(n-30) \bmod 5+1}(3), P(n-18,8)\big)$
$\big(R_\infty^{(n-29) \bmod 2+1}(1), P(n-15,8)\big),\ \big(R_1^{(n-29) \bmod 5+1}(2), P(n-12,8)\big),\ \big(R_0^{(n-30) \bmod 3+1}(1), P(n-10,8)\big)$
$\big(R_1^{(n-28) \bmod 5+1}(1), P(n-6,8)\big),\ \big(uI, (u+1)P\big),\ n > 30$

Modules of the form $P(n,9)$

Defect: $\partial P(n,9) = -3$, for $n \geq 0$.

$P(0,9) : \big(I(0,9), P(0,3)\big)$

$P(1,9) : \big(I(4,8), P(0,2)\big),\ \big(I(1,1), P(0,4)\big)$

$P(2,9) : \big(P(3,8), P(0,1)\big),\ \big(I(5,8), P(0,5)\big),\ \big(R_1^5(1), P(0,9)\big),\ \big(I(3,8), P(1,2)\big),\ \big(I(0,1), P(1,4)\big)$

$P(3,9) : \big(R_1^4(1), P(0,6)\big),\ \big(P(4,8), P(1,1)\big),\ \big(I(4,8), P(1,5)\big),\ \big(R_1^1(1), P(1,9)\big),\ \big(I(2,8), P(2,2)\big)$

$P(4,9) : \big(P(9,8), P(0,1)\big),\ \big(P(7,8), P(0,7)\big),\ \big(R_1^5(1), P(1,6)\big),\ \big(P(5,8), P(2,1)\big),\ \big(I(3,8), P(2,5)\big)$
$\big(R_1^2(1), P(2,9)\big),\ \big(I(1,8), P(3,2)\big)$

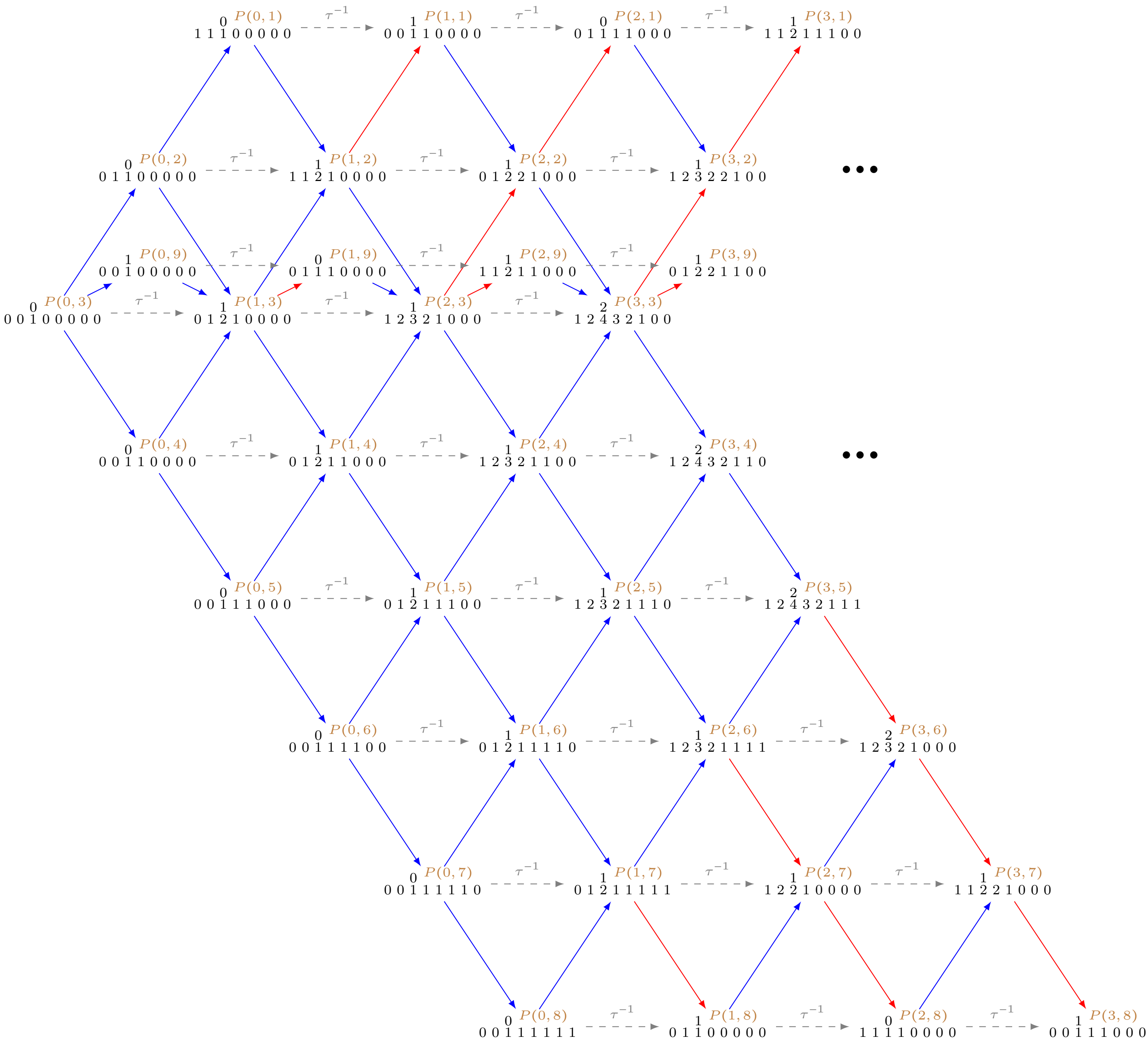
$P(5,9) : \big(P(6,1), P(0,8)\big),\ \big(P(10,8), P(1,1)\big),\ \big(P(8,8), P(1,7)\big),\ \big(R_1^1(1), P(2,6)\big),\ \big(P(6,8), P(3,1)\big)$
$\big(I(2,8), P(3,5)\big),\ \big(R_1^3(1), P(3,9)\big),\ \big(I(0,8), P(4,2)\big)$

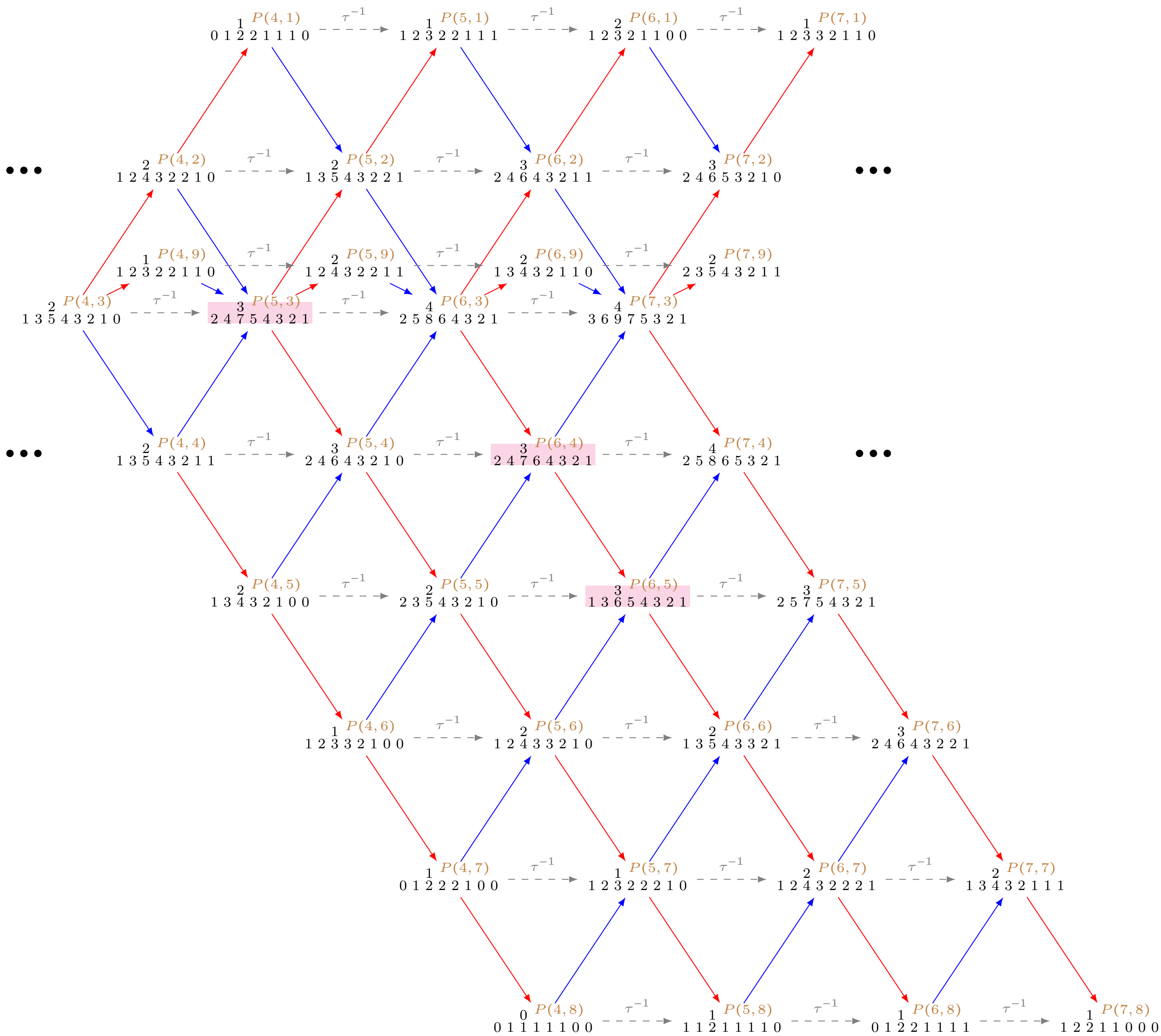
$P(6,9) : \big(P(7,1), P(1,8)\big),\ \big(P(11,8), P(2,1)\big),\ \big(P(9,8), P(2,7)\big),\ \big(R_1^2(1), P(3,6)\big),\ \big(P(7,8), P(4,1)\big)$
$\big(I(1,8), P(4,5)\big),\ \big(R_1^4(1), P(4,9)\big)$

$P(7,9) : \big(P(8,7), P(0,8)\big),\ \big(P(8,1), P(2,8)\big),\ \big(P(12,8), P(3,1)\big),\ \big(P(10,8), P(3,7)\big),\ \big(R_1^3(1), P(4,6)\big)$
$\big(P(8,8), P(5,1)\big),\ \big(I(0,8), P(5,5)\big),\ \big(R_1^5(1), P(5,9)\big)$

$P(8,9) : \big(P(9,7), P(1,8)\big),\ \big(P(9,1), P(3,8)\big),\ \big(P(13,8), P(4,1)\big),\ \big(P(11,8), P(4,7)\big),\ \big(R_1^4(1), P(5,6)\big)$
$\big(P(9,8), P(6,1)\big),\ \big(R_1^1(1), P(6,9)\big)$

$$P(9,9):\big(P(12,1),P(0,8)\big),\ \big(P(10,7),P(2,8)\big),\ \big(P(10,1),P(4,8)\big),\ \big(P(14,8),P(5,1)\big),\ \big(P(12,8),P(5,7)\big)$$
$$\big(R_1^5(1),P(6,6)\big),\ \big(P(10,8),P(7,1)\big),\ \big(R_1^2(1),P(7,9)\big)$$
$$P(n,9):\big(P(n+3,1),P(n-9,8)\big),\ \big(P(n+1,7),P(n-7,8)\big),\ \big(P(n+1,1),P(n-5,8)\big)$$
$$\big(P(n+5,8),P(n-4,1)\big),\ \big(P(n+3,8),P(n-4,7)\big),\ \big(R_1^{(n-5)\text{ mod }5+1}(1),P(n-3,6)\big)$$
$$\big(P(n+1,8),P(n-2,1)\big),\ \big(R_1^{(n-8)\text{ mod }5+1}(1),P(n-2,9)\big),\ n>9$$




P(4,1)
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P(6,1)
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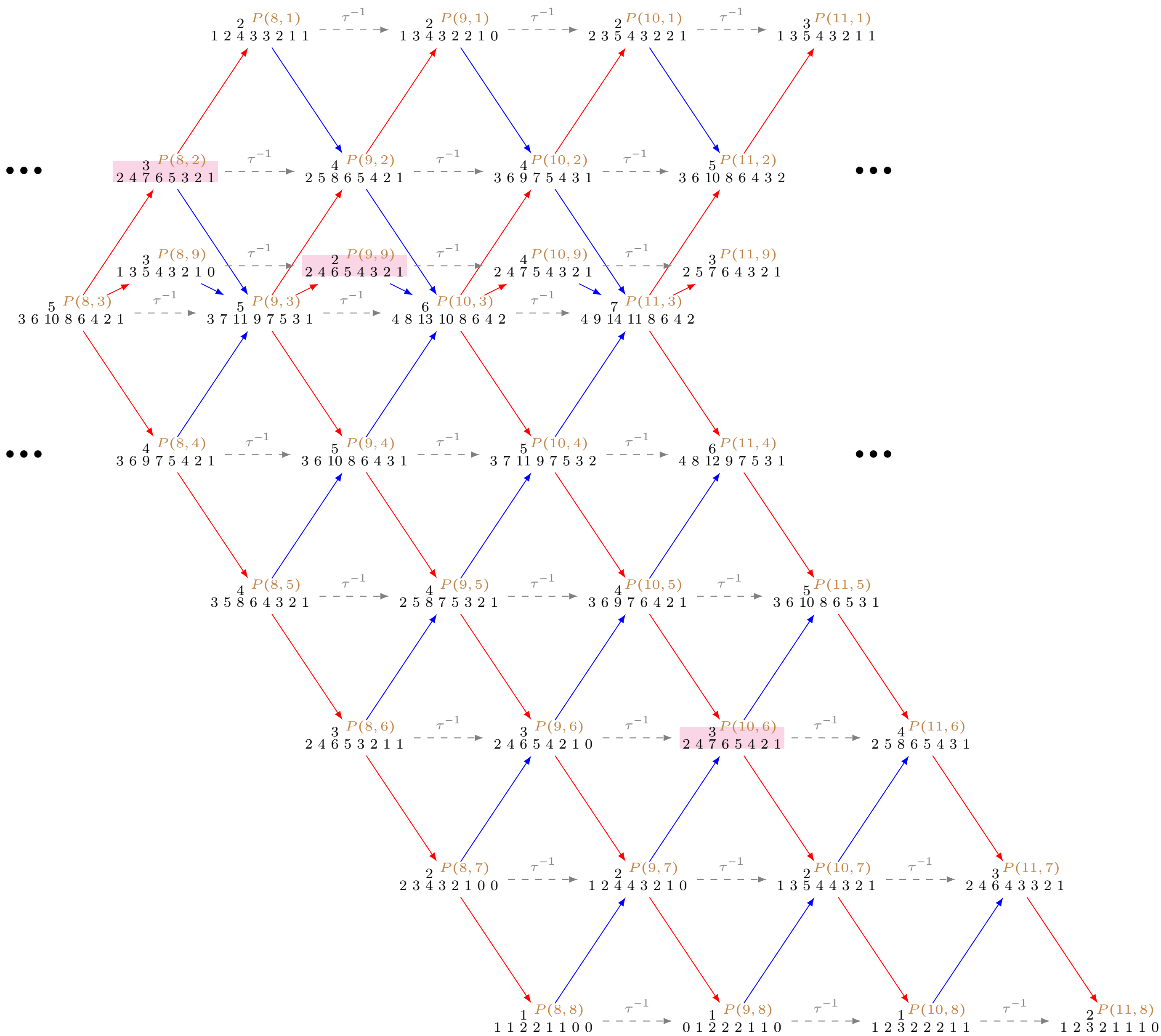
P(7,8)
τ^{-1}


P(8,1)
P(9,1)
P(10,1)
P(11,1)
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P(11,2)
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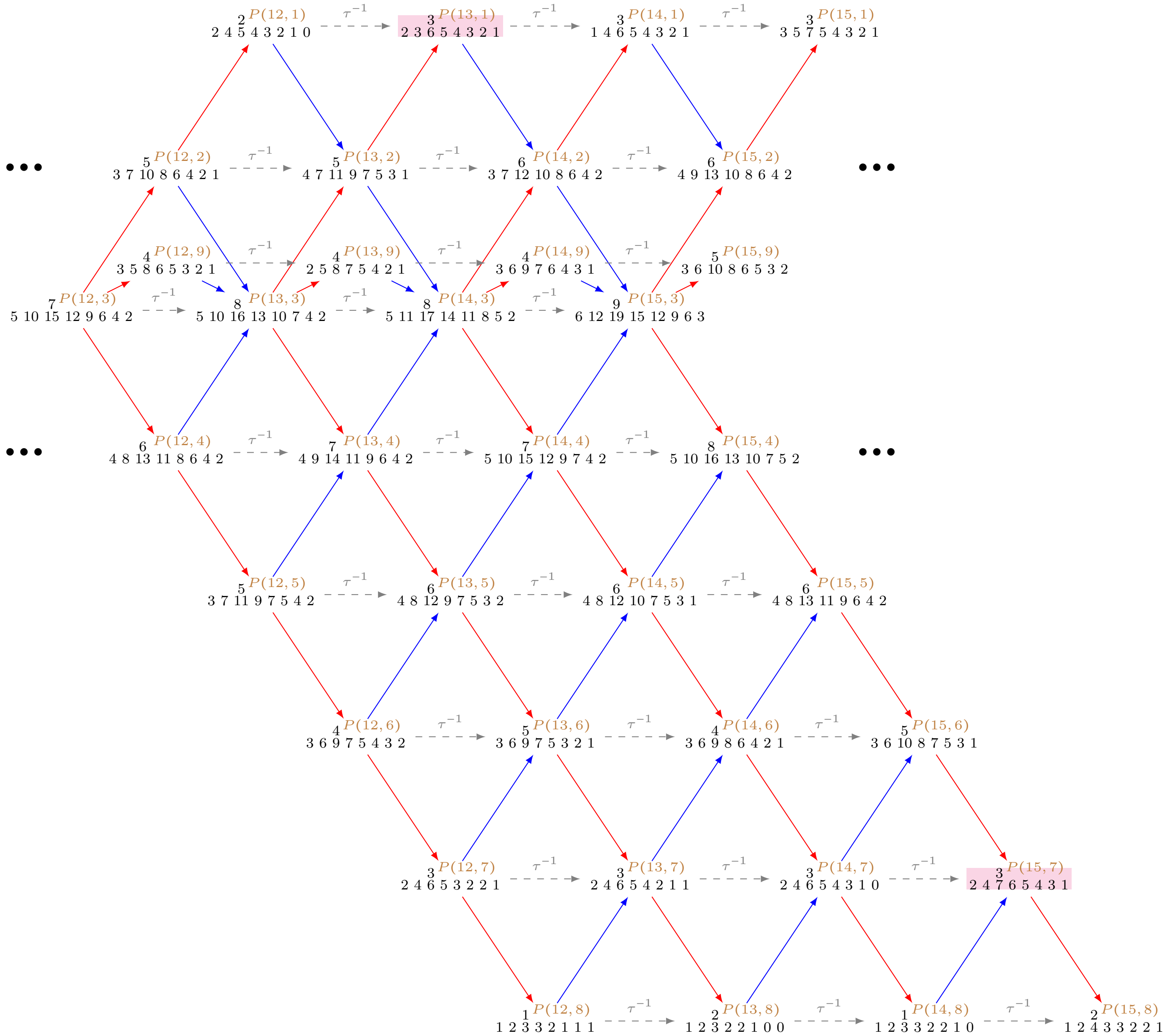
P(10,8)
P(11,8)
τ^{-1}


P(12,1)
P(13,1)
P(14,1)
P(15,1)
P(12,2)
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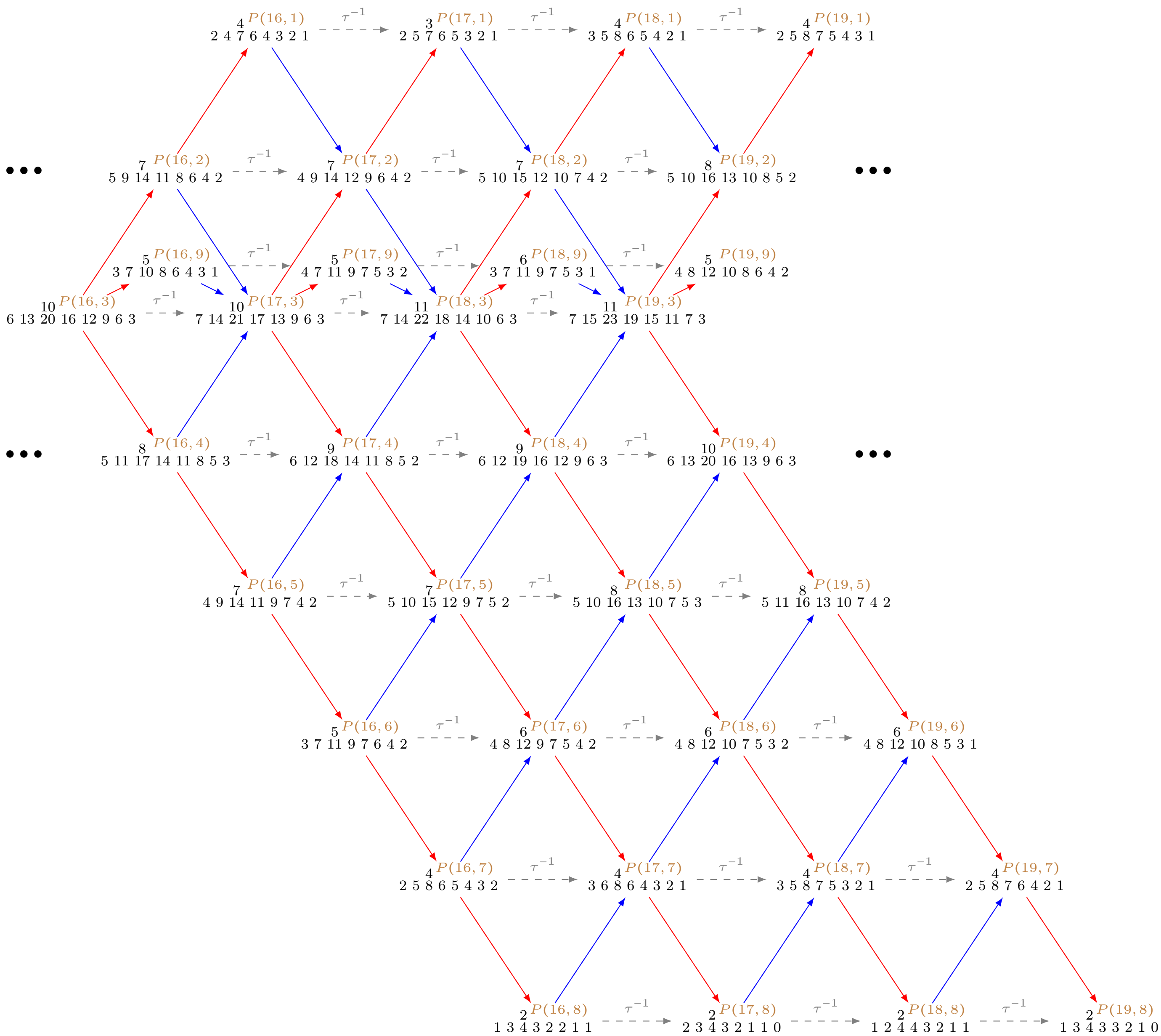

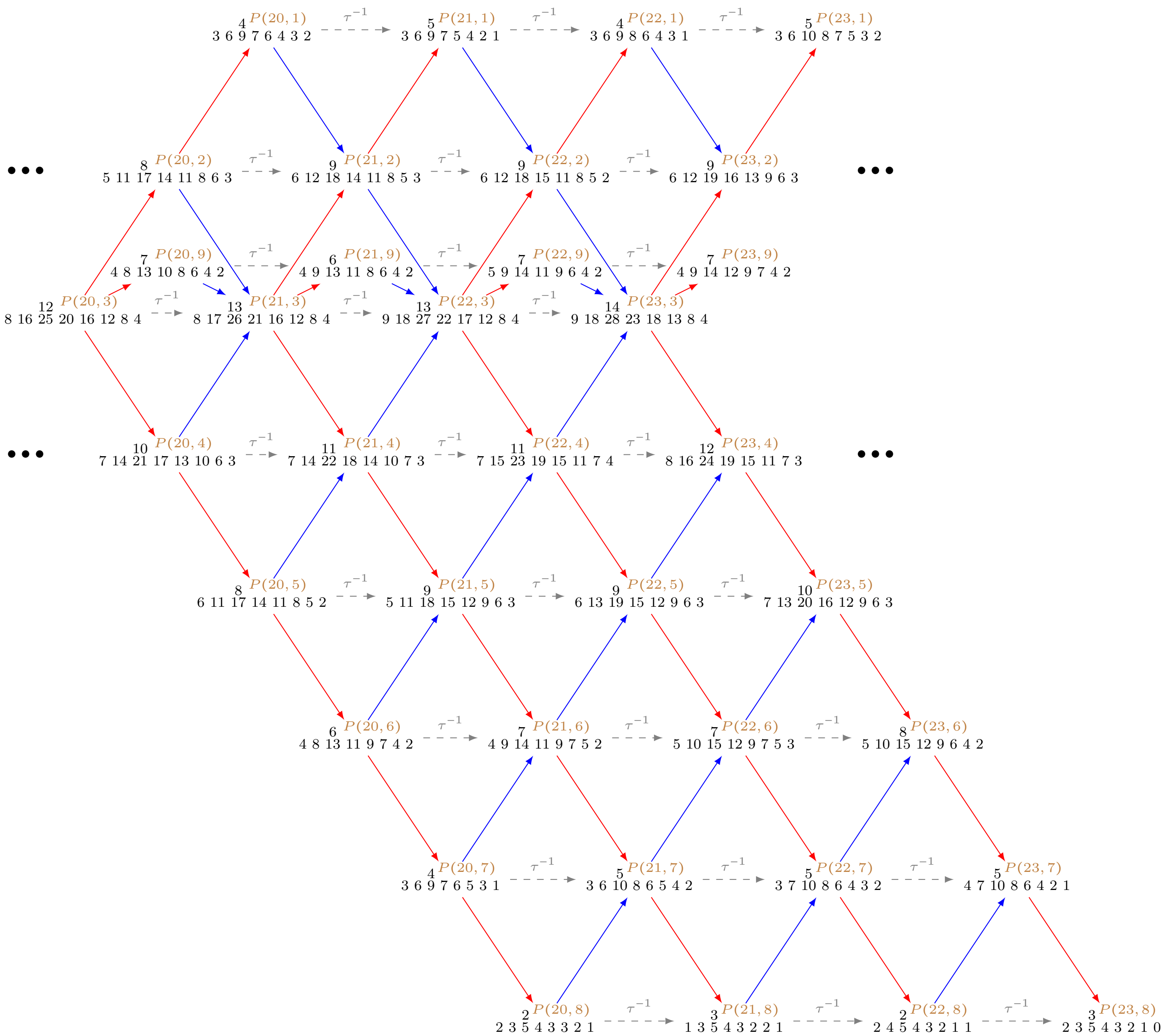

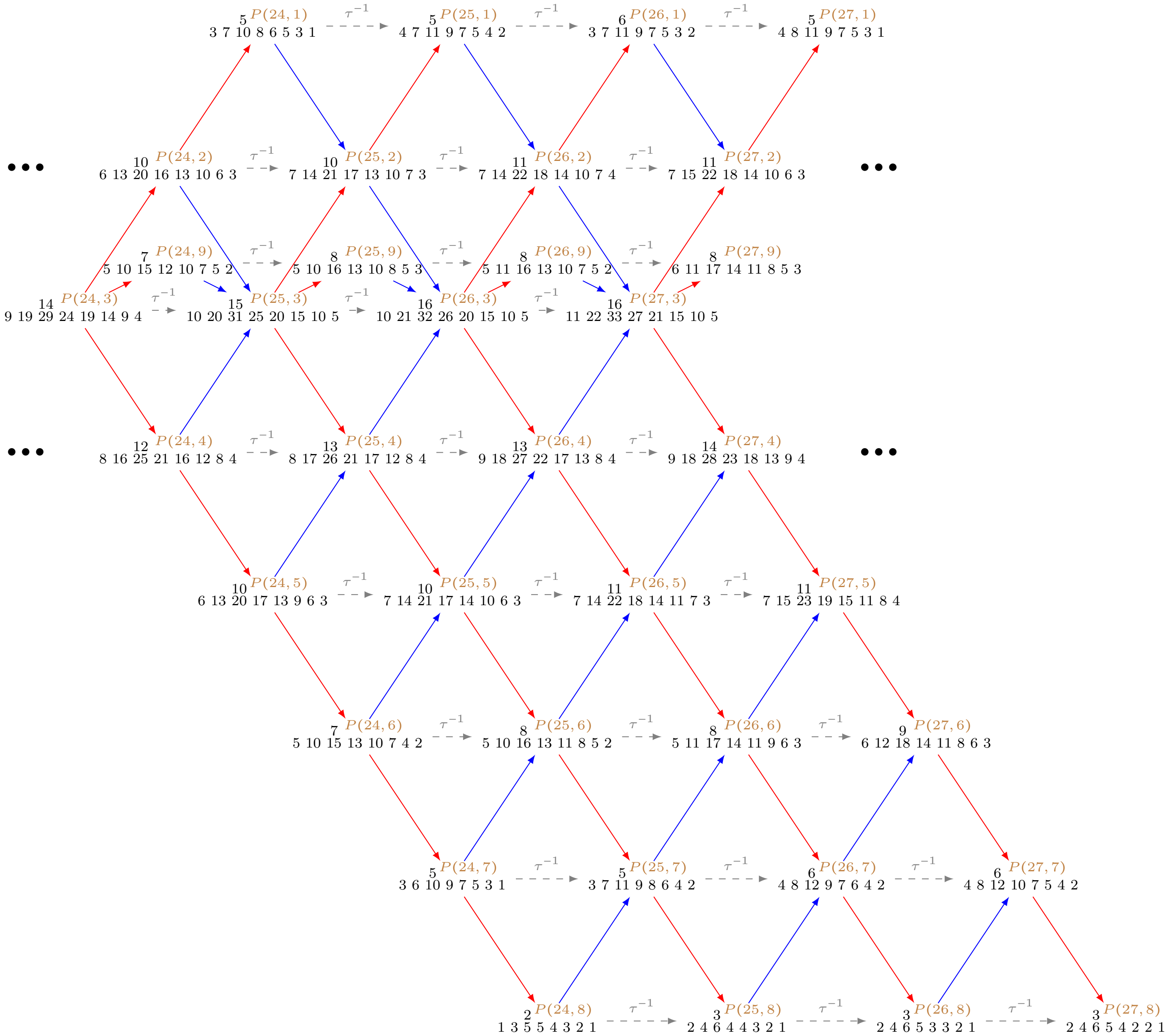

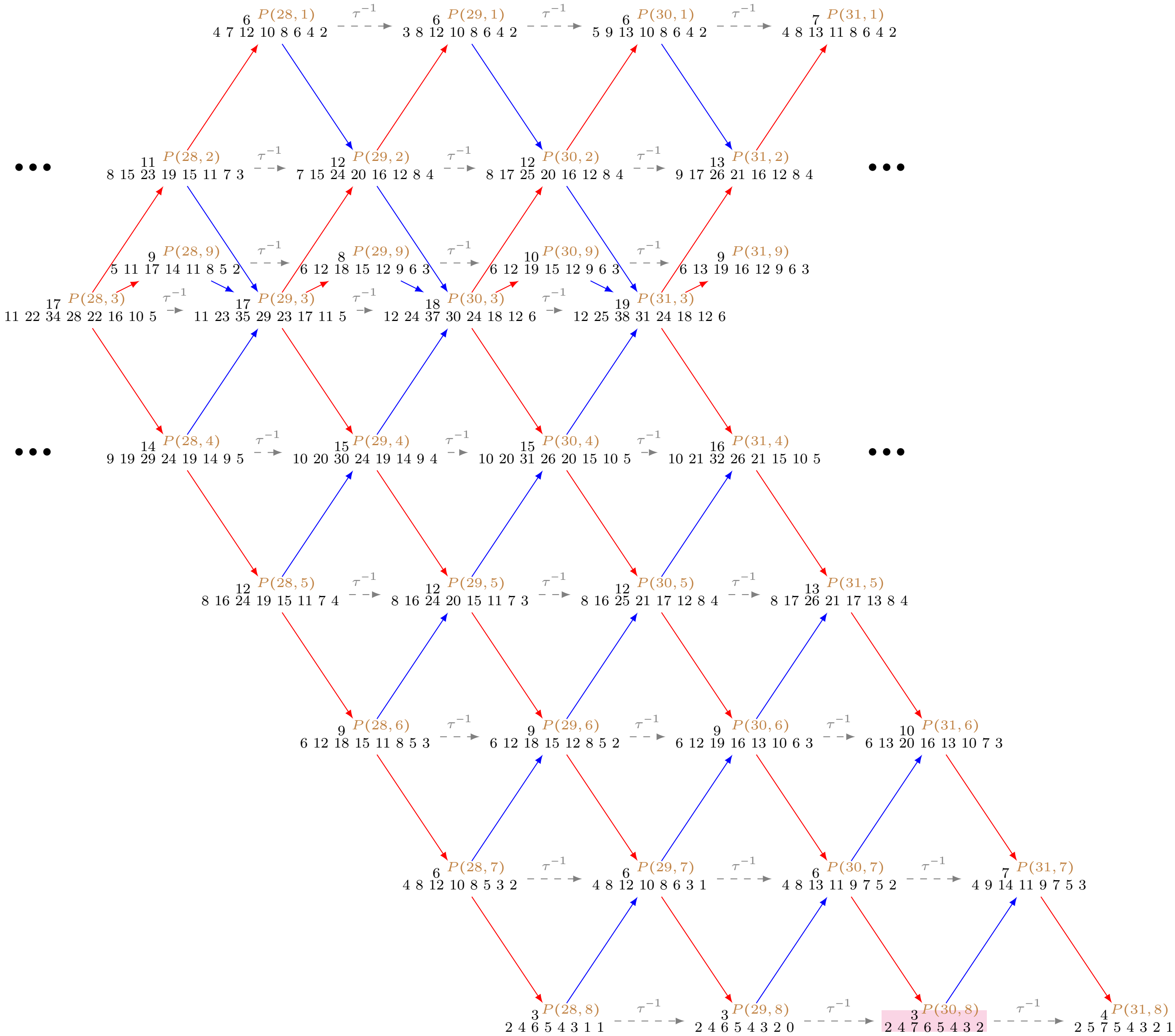
P(28,1)
P(29,1)
P(30,1)
P(31,1)
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P(31,2)
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P(30,8)
P(31,8)
τ^{-1}

Schofield pairs associated to preinjective exceptional modules

Modules of the form $I(n,1)$

Defect: $\partial I(n,1) = 2$, for $n \geq 0$.

$I(0,1):-$

$I(1,1):-$

$I(2,1):\big(I(0,4),P(0,9)\big),\ \big(I(0,5),P(1,1)\big),\ \big(I(0,6),P(3,8)\big),\ \big(I(0,7),R_1^1(1)\big),\ \big(I(0,8),I(7,8)\big)$
$\big(I(0,9),P(0,8)\big)$

$I(3,1):\big(I(0,1),R_1^2(1)\big),\ \big(I(0,2),P(0,7)\big),\ \big(I(1,5),P(0,1)\big),\ \big(I(1,6),P(2,8)\big),\ \big(I(1,7),R_1^5(1)\big)$
$\big(I(1,8),I(8,8)\big)$

$I(4,1):\big(I(0,9),P(4,8)\big),\ \big(I(1,1),R_1^1(1)\big),\ \big(I(2,6),P(1,8)\big),\ \big(I(2,7),R_1^4(1)\big),\ \big(I(2,8),I(9,8)\big)$

$I(5,1):\big(I(0,1),R_0^1(1)\big),\ \big(I(0,7),R_1^5(2)\big),\ \big(I(0,8),I(13,8)\big),\ \big(I(1,9),P(3,8)\big),\ \big(I(2,1),R_1^5(1)\big)$
$\big(I(3,6),P(0,8)\big),\ \big(I(3,7),R_1^3(1)\big),\ \big(I(3,8),I(10,8)\big)$

$I(6,1):\big(I(1,1),R_0^3(1)\big),\ \big(I(1,7),R_1^4(2)\big),\ \big(I(1,8),I(14,8)\big),\ \big(I(2,9),P(2,8)\big),\ \big(I(3,1),R_1^4(1)\big)$
$\big(I(4,7),R_1^2(1)\big),\ \big(I(4,8),I(11,8)\big)$

$I(7,1):\big(I(0,8),I(17,8)\big),\ \big(I(2,1),R_0^2(1)\big),\ \big(I(2,7),R_1^3(2)\big),\ \big(I(2,8),I(15,8)\big),\ \big(I(3,9),P(1,8)\big)$
$\big(I(4,1),R_1^3(1)\big),\ \big(I(5,7),R_1^1(1)\big),\ \big(I(5,8),I(12,8)\big)$

$I(8,1):\big(I(1,8),I(18,8)\big),\ \big(I(3,1),R_0^1(1)\big),\ \big(I(3,7),R_1^2(2)\big),\ \big(I(3,8),I(16,8)\big),\ \big(I(4,9),P(0,8)\big)$
$\big(I(5,1),R_1^2(1)\big),\ \big(I(6,7),R_1^5(1)\big),\ \big(I(6,8),I(13,8)\big)$

$I(9,1):\big(I(2,8),I(19,8)\big),\ \big(I(4,1),R_0^3(1)\big),\ \big(I(4,7),R_1^1(2)\big),\ \big(I(4,8),I(17,8)\big),\ \big(I(6,1),R_1^1(1)\big)$
$\big(I(7,7),R_1^4(1)\big),\ \big(I(7,8),I(14,8)\big)$

$I(10,1):\big(I(0,8),I(23,8)\big),\ \big(I(3,8),I(20,8)\big),\ \big(I(5,1),R_0^2(1)\big),\ \big(I(5,7),R_1^5(2)\big),\ \big(I(5,8),I(18,8)\big)$
$\big(I(7,1),R_1^5(1)\big),\ \big(I(8,7),R_1^3(1)\big),\ \big(I(8,8),I(15,8)\big)$

$I(n,1):\big(I(n-10,8),I(n+13,8)\big),\ \big(I(n-7,8),I(n+10,8)\big),\ \big(I(n-5,1),R_0^{(-n+11) \bmod 3+1}(1)\big)$
$\big(I(n-5,7),R_1^{(-n+14) \bmod 5+1}(2)\big),\ \big(I(n-5,8),I(n+8,8)\big),\ \big(I(n-3,1),R_1^{(-n+14) \bmod 5+1}(1)\big)$
$\big(I(n-2,7),R_1^{(-n+12) \bmod 5+1}(1)\big),\ \big(I(n-2,8),I(n+5,8)\big),\ n>10$

Modules of the form $I(n,2)$

Defect: $\partial I(n,2) = 4$, for $n \geq 0$.

$I(0,2):\big(I(0,1),I(1,1)\big)$

$I(1,2):\big(I(0,4),P(1,8)\big),\ \big(I(0,5),R_1^4(1)\big),\ \big(I(0,6),I(9,8)\big),\ \big(I(0,7),I(4,1)\big),\ \big(I(0,8),I(2,9)\big)$
$\big(I(0,9),I(6,8)\big),\ \big(I(1,1),I(2,1)\big)$

$I(2,2):\big(I(0,1),I(6,7)\big),\ \big(I(1,4),P(0,8)\big),\ \big(I(1,5),R_1^3(1)\big),\ \big(I(1,6),I(10,8)\big),\ \big(I(1,7),I(5,1)\big)$
$\big(I(1,8),I(3,9)\big),\ \big(I(1,9),I(7,8)\big),\ \big(I(2,1),I(3,1)\big)$

$I(3,2) : \big(I(1,1), I(7,7)\big),\ \big(I(2,5), R_1^2(1)\big),\ \big(I(2,6), I(11,8)\big),\ \big(I(2,7), I(6,1)\big),\ \big(I(2,8), I(4,9)\big)$
$\big(I(2,9), I(8,8)\big),\ \big(I(3,1), I(4,1)\big)$
$I(4,2) : \big(I(0,8), I(7,6)\big),\ \big(I(2,1), I(8,7)\big),\ \big(I(3,5), R_1^1(1)\big),\ \big(I(3,6), I(12,8)\big),\ \big(I(3,7), I(7,1)\big)$
$\big(I(3,8), I(5,9)\big),\ \big(I(3,9), I(9,8)\big),\ \big(I(4,1), I(5,1)\big)$
$I(n,2) : \big(I(n-4,8), I(n+3,6)\big),\ \big(I(n-2,1), I(n+4,7)\big),\ \big(I(n-1,5), R_1^{(-n+4) \bmod 5+1}(1)\big)$
$\big(I(n-1,6), I(n+8,8)\big),\ \big(I(n-1,7), I(n+3,1)\big),\ \big(I(n-1,8), I(n+1,9)\big)$
$\big(I(n-1,9), I(n+5,8)\big),\ \big(I(n,1), I(n+1,1)\big),\ n > 4$

Modules of the form $I(n,3)$

Defect: $\partial I(n,3) = 6$, for $n \geq 0$.

$I(0,3) : \big(I(0,1), I(1,2)\big),\ \big(I(0,2), I(2,1)\big),\ \big(I(0,4), I(5,8)\big),\ \big(I(0,5), I(4,7)\big),\ \big(I(0,6), I(3,6)\big)$
$\big(I(0,7), I(2,5)\big),\ \big(I(0,8), I(1,4)\big),\ \big(I(0,9), I(1,9)\big)$
$I(n,3) : \big(I(n,1), I(n+1,2)\big),\ \big(I(n,2), I(n+2,1)\big),\ \big(I(n,4), I(n+5,8)\big)$
$\big(I(n,5), I(n+4,7)\big),\ \big(I(n,6), I(n+3,6)\big),\ \big(I(n,7), I(n+2,5)\big)$
$\big(I(n,8), I(n+1,4)\big),\ \big(I(n,9), I(n+1,9)\big),\ n > 0$

Modules of the form $I(n,4)$

Defect: $\partial I(n,4) = 5$, for $n \geq 0$.

$I(0,4) : \big(I(0,5), I(4,8)\big),\ \big(I(0,6), I(3,7)\big),\ \big(I(0,7), I(2,6)\big),\ \big(I(0,8), I(1,5)\big)$
$I(1,4) : \big(I(0,1), I(2,9)\big),\ \big(I(0,2), I(7,8)\big),\ \big(I(0,9), I(3,1)\big),\ \big(I(1,5), I(5,8)\big),\ \big(I(1,6), I(4,7)\big)$
$\big(I(1,7), I(3,6)\big),\ \big(I(1,8), I(2,5)\big)$
$I(2,4) : \big(I(0,8), I(3,2)\big),\ \big(I(1,1), I(3,9)\big),\ \big(I(1,2), I(8,8)\big),\ \big(I(1,9), I(4,1)\big),\ \big(I(2,5), I(6,8)\big)$
$\big(I(2,6), I(5,7)\big),\ \big(I(2,7), I(4,6)\big),\ \big(I(2,8), I(3,5)\big)$
$I(n,4) : \big(I(n-2,8), I(n+1,2)\big),\ \big(I(n-1,1), I(n+1,9)\big),\ \big(I(n-1,2), I(n+6,8)\big)$
$\big(I(n-1,9), I(n+2,1)\big),\ \big(I(n,5), I(n+4,8)\big),\ \big(I(n,6), I(n+3,7)\big)$
$\big(I(n,7), I(n+2,6)\big),\ \big(I(n,8), I(n+1,5)\big),\ n > 2$

Modules of the form $I(n,5)$

Defect: $\partial I(n,5) = 4$, for $n \geq 0$.

$I(0,5) : \big(I(0,6), I(3,8)\big),\ \big(I(0,7), I(2,7)\big),\ \big(I(0,8), I(1,6)\big)$
$I(1,5) : \big(I(1,6), I(4,8)\big),\ \big(I(1,7), I(3,7)\big),\ \big(I(1,8), I(2,6)\big)$

$I(2,5):\big(I(0,1),I(4,1)\big),\ \big(I(0,2),R_1^1(1)\big),\ \big(I(0,9),I(8,8)\big),\ \big(I(2,6),I(5,8)\big),\ \big(I(2,7),I(4,7)\big)$
$\big(I(2,8),I(3,6)\big)$

$I(3,5):\big(I(0,7),I(8,7)\big),\ \big(I(0,8),I(4,9)\big),\ \big(I(1,1),I(5,1)\big),\ \big(I(1,2),R_1^5(1)\big),\ \big(I(1,9),I(9,8)\big)$
$\big(I(3,6),I(6,8)\big),\ \big(I(3,7),I(5,7)\big),\ \big(I(3,8),I(4,6)\big)$

$I(n,5):\big(I(n-3,7),I(n+5,7)\big),\ \big(I(n-3,8),I(n+1,9)\big),\ \big(I(n-2,1),I(n+2,1)\big)$
$\big(I(n-2,2),R_1^{(-n+7)\bmod 5+1}(1)\big),\ \big(I(n-2,9),I(n+6,8)\big),\ \big(I(n,6),I(n+3,8)\big)$
$\big(I(n,7),I(n+2,7)\big),\ \big(I(n,8),I(n+1,6)\big),\ n>3$

Modules of the form $I(n,6)$

Defect: $\partial I(n,6)=3$, for $n\geq 0$.

$I(0,6):\big(I(0,7),I(2,8)\big),\ \big(I(0,8),I(1,7)\big)$

$I(1,6):\big(I(1,7),I(3,8)\big),\ \big(I(1,8),I(2,7)\big)$

$I(2,6):\big(I(2,7),I(4,8)\big),\ \big(I(2,8),I(3,7)\big)$

$I(3,6):\big(I(0,1),I(9,8)\big),\ \big(I(0,2),P(3,8)\big),\ \big(I(0,9),R_1^5(1)\big),\ \big(I(3,7),I(5,8)\big),\ \big(I(3,8),I(4,7)\big)$

$I(4,6):\big(I(0,6),R_1^4(2)\big),\ \big(I(0,7),I(14,8)\big),\ \big(I(0,8),I(6,1)\big),\ \big(I(1,1),I(10,8)\big),\ \big(I(1,2),P(2,8)\big)$
$\big(I(1,9),R_1^4(1)\big),\ \big(I(4,7),I(6,8)\big),\ \big(I(4,8),I(5,7)\big)$

$I(5,6):\big(I(1,6),R_1^3(2)\big),\ \big(I(1,7),I(15,8)\big),\ \big(I(1,8),I(7,1)\big),\ \big(I(2,1),I(11,8)\big),\ \big(I(2,2),P(1,8)\big)$
$\big(I(2,9),R_1^3(1)\big),\ \big(I(5,7),I(7,8)\big),\ \big(I(5,8),I(6,7)\big)$

$I(6,6):\big(I(2,6),R_1^2(2)\big),\ \big(I(2,7),I(16,8)\big),\ \big(I(2,8),I(8,1)\big),\ \big(I(3,1),I(12,8)\big),\ \big(I(3,2),P(0,8)\big)$
$\big(I(3,9),R_1^2(1)\big),\ \big(I(6,7),I(8,8)\big),\ \big(I(6,8),I(7,7)\big)$

$I(7,6):\big(I(3,6),R_1^1(2)\big),\ \big(I(3,7),I(17,8)\big),\ \big(I(3,8),I(9,1)\big),\ \big(I(4,1),I(13,8)\big),\ \big(I(4,9),R_1^1(1)\big)$
$\big(I(7,7),I(9,8)\big),\ \big(I(7,8),I(8,7)\big)$

$I(8,6):\big(I(0,8),I(13,7)\big),\ \big(I(4,6),R_1^5(2)\big),\ \big(I(4,7),I(18,8)\big),\ \big(I(4,8),I(10,1)\big),\ \big(I(5,1),I(14,8)\big)$
$\big(I(5,9),R_1^5(1)\big),\ \big(I(8,7),I(10,8)\big),\ \big(I(8,8),I(9,7)\big)$

$I(n,6):\big(I(n-8,8),I(n+5,7)\big),\ \big(I(n-4,6),R_1^{(-n+12)\bmod 5+1}(2)\big),\ \big(I(n-4,7),I(n+10,8)\big)$
$\big(I(n-4,8),I(n+2,1)\big),\ \big(I(n-3,1),I(n+6,8)\big),\ \big(I(n-3,9),R_1^{(-n+12)\bmod 5+1}(1)\big)$
$\big(I(n,7),I(n+2,8)\big),\ \big(I(n,8),I(n+1,7)\big),\ n>8$

Modules of the form $I(n,7)$

Defect: $\partial I(n,7)=2$, for $n\geq 0$.

$I(0,7):\big(I(0,8),I(1,8)\big)$

$I(1,7):\big(I(1,8),I(2,8)\big)$

$I(2,7) : \big(I(2,8), I(3,8)\big)$
$I(3,7) : \big(I(3,8), I(4,8)\big)$
$I(4,7) : \big(I(0,1), R_1^4(1)\big),\ \big(I(0,2), P(1,1)\big),\ \big(I(0,9), P(2,8)\big),\ \big(I(4,8), I(5,8)\big)$
$I(5,7) : \big(I(0,5), P(2,7)\big),\ \big(I(0,6), P(7,8)\big),\ \big(I(0,7), R_0^2(1)\big),\ \big(I(0,8), I(11,8)\big),\ \big(I(1,1), R_1^3(1)\big)$
$\big(I(1,2), P(0,1)\big),\ \big(I(1,9), P(1,8)\big),\ \big(I(5,8), I(6,8)\big)$
$I(6,7) : \big(I(1,5), P(1,7)\big),\ \big(I(1,6), P(6,8)\big),\ \big(I(1,7), R_0^1(1)\big),\ \big(I(1,8), I(12,8)\big),\ \big(I(2,1), R_1^2(1)\big)$
$\big(I(2,9), P(0,8)\big),\ \big(I(6,8), I(7,8)\big)$
$I(7,7) : \big(I(0,1), R_1^1(2)\big),\ \big(I(2,5), P(0,7)\big),\ \big(I(2,6), P(5,8)\big),\ \big(I(2,7), R_0^3(1)\big),\ \big(I(2,8), I(13,8)\big)$
$\big(I(3,1), R_1^1(1)\big),\ \big(I(7,8), I(8,8)\big)$
$I(8,7) : \big(I(1,1), R_1^5(2)\big),\ \big(I(3,6), P(4,8)\big),\ \big(I(3,7), R_0^2(1)\big),\ \big(I(3,8), I(14,8)\big),\ \big(I(4,1), R_1^5(1)\big)$
$\big(I(8,8), I(9,8)\big)$
$I(9,7) : \big(I(0,7), R_1^4(3)\big),\ \big(I(0,8), I(19,8)\big),\ \big(I(2,1), R_1^4(2)\big),\ \big(I(4,6), P(3,8)\big),\ \big(I(4,7), R_0^1(1)\big)$
$\big(I(4,8), I(15,8)\big),\ \big(I(5,1), R_1^4(1)\big),\ \big(I(9,8), I(10,8)\big)$
$I(10,7) : \big(I(1,7), R_1^3(3)\big),\ \big(I(1,8), I(20,8)\big),\ \big(I(3,1), R_1^3(2)\big),\ \big(I(5,6), P(2,8)\big),\ \big(I(5,7), R_0^3(1)\big)$
$\big(I(5,8), I(16,8)\big),\ \big(I(6,1), R_1^3(1)\big),\ \big(I(10,8), I(11,8)\big)$
$I(11,7) : \big(I(2,7), R_1^2(3)\big),\ \big(I(2,8), I(21,8)\big),\ \big(I(4,1), R_1^2(2)\big),\ \big(I(6,6), P(1,8)\big),\ \big(I(6,7), R_0^2(1)\big)$
$\big(I(6,8), I(17,8)\big),\ \big(I(7,1), R_1^2(1)\big),\ \big(I(11,8), I(12,8)\big)$
$I(12,7) : \big(I(3,7), R_1^1(3)\big),\ \big(I(3,8), I(22,8)\big),\ \big(I(5,1), R_1^1(2)\big),\ \big(I(7,6), P(0,8)\big),\ \big(I(7,7), R_0^1(1)\big)$
$\big(I(7,8), I(18,8)\big),\ \big(I(8,1), R_1^1(1)\big),\ \big(I(12,8), I(13,8)\big)$
$I(13,7) : \big(I(4,7), R_1^5(3)\big),\ \big(I(4,8), I(23,8)\big),\ \big(I(6,1), R_1^5(2)\big),\ \big(I(8,7), R_0^3(1)\big),\ \big(I(8,8), I(19,8)\big)$
$\big(I(9,1), R_1^5(1)\big),\ \big(I(13,8), I(14,8)\big)$
$I(14,7) : \big(I(0,8), I(29,8)\big),\ \big(I(5,7), R_1^4(3)\big),\ \big(I(5,8), I(24,8)\big),\ \big(I(7,1), R_1^4(2)\big),\ \big(I(9,7), R_0^2(1)\big)$
$\big(I(9,8), I(20,8)\big),\ \big(I(10,1), R_1^4(1)\big),\ \big(I(14,8), I(15,8)\big)$
$I(n,7) : \big(I(n-14,8), I(n+15,8)\big),\ \big(I(n-9,7), R_1^{(-n+17) \bmod 5+1}(3)\big),\ \big(I(n-9,8), I(n+10,8)\big)$
$\big(I(n-7,1), R_1^{(-n+17) \bmod 5+1}(2)\big),\ \big(I(n-5,7), R_0^{(-n+15) \bmod 3+1}(1)\big),\ \big(I(n-5,8), I(n+6,8)\big)$
$\big(I(n-4,1), R_1^{(-n+17) \bmod 5+1}(1)\big),\ \big(I(n,8), I(n+1,8)\big),\ n > 14$

Modules of the form $I(n,8)$

Defect: $\partial I(n,8) = 1$, for $n \geq 0$.

$I(0,8) : -$
$I(1,8) : -$
$I(2,8) : -$
$I(3,8) : -$

$I(4,8)$: –

$I(5,8)$: $(I(0,1),P(1,8))$, $(I(0,2),P(0,9))$, $(I(0,9),P(0,1))$

$I(6,8)$: $(I(0,4),P(0,2))$, $(I(0,5),P(1,9))$, $(I(0,6),P(2,1))$, $(I(0,7),P(4,8))$, $(I(0,8),R_1^2(1))$ $(I(1,1),P(0,8))$

$I(7,8)$: $(I(0,9),P(0,7))$, $(I(1,5),P(0,9))$, $(I(1,6),P(1,1))$, $(I(1,7),P(3,8))$, $(I(1,8),R_1^1(1))$

$I(8,8)$: $(I(0,1),P(4,8))$, $(I(0,2),P(0,6))$, $(I(2,6),P(0,1))$, $(I(2,7),P(2,8))$, $(I(2,8),R_1^5(1))$

$I(9,8)$: $(I(0,9),P(2,1))$, $(I(1,1),P(3,8))$, $(I(3,7),P(1,8))$, $(I(3,8),R_1^4(1))$

$I(10,8)$: $(I(0,1),P(6,8))$, $(I(0,6),P(3,7))$, $(I(0,7),P(8,8))$, $(I(0,8),R_0^3(1))$, $(I(1,9),P(1,1))$ $(I(2,1),P(2,8))$, $(I(4,7),P(0,8))$, $(I(4,8),R_1^3(1))$

$I(11,8)$: $(I(1,1),P(5,8))$, $(I(1,6),P(2,7))$, $(I(1,7),P(7,8))$, $(I(1,8),R_0^2(1))$, $(I(2,9),P(0,1))$ $(I(3,1),P(1,8))$, $(I(5,8),R_1^2(1))$

$I(12,8)$: $(I(0,8),R_1^1(2))$, $(I(2,1),P(4,8))$, $(I(2,6),P(1,7))$, $(I(2,7),P(6,8))$, $(I(2,8),R_0^1(1))$ $(I(4,1),P(0,8))$, $(I(6,8),R_1^1(1))$

$I(13,8)$: $(I(0,1),P(9,8))$, $(I(1,8),R_1^5(2))$, $(I(3,1),P(3,8))$, $(I(3,6),P(0,7))$, $(I(3,7),P(5,8))$ $(I(3,8),R_0^3(1))$, $(I(7,8),R_1^5(1))$

$I(14,8)$: $(I(1,1),P(8,8))$, $(I(2,8),R_1^4(2))$, $(I(4,1),P(2,8))$, $(I(4,7),P(4,8))$, $(I(4,8),R_0^2(1))$ $(I(8,8),R_1^4(1))$

$I(15,8)$: $(I(0,7),P(13,8))$, $(I(0,8),R_\infty^1(1))$, $(I(2,1),P(7,8))$, $(I(3,8),R_1^3(2))$, $(I(5,1),P(1,8))$ $(I(5,7),P(3,8))$, $(I(5,8),R_0^1(1))$, $(I(9,8),R_1^3(1))$

$I(16,8)$: $(I(1,7),P(12,8))$, $(I(1,8),R_\infty^2(1))$, $(I(3,1),P(6,8))$, $(I(4,8),R_1^2(2))$, $(I(6,1),P(0,8))$ $(I(6,7),P(2,8))$, $(I(6,8),R_0^3(1))$, $(I(10,8),R_1^2(1))$

$I(17,8)$: $(I(2,7),P(11,8))$, $(I(2,8),R_\infty^1(1))$, $(I(4,1),P(5,8))$, $(I(5,8),R_1^1(2))$, $(I(7,7),P(1,8))$ $(I(7,8),R_0^2(1))$, $(I(11,8),R_1^1(1))$

$I(18,8)$: $(I(0,8),R_1^5(3))$, $(I(3,7),P(10,8))$, $(I(3,8),R_\infty^2(1))$, $(I(5,1),P(4,8))$, $(I(6,8),R_1^5(2))$ $(I(8,7),P(0,8))$, $(I(8,8),R_0^1(1))$, $(I(12,8),R_1^5(1))$

$I(19,8)$: $(I(1,8),R_1^4(3))$, $(I(4,7),P(9,8))$, $(I(4,8),R_\infty^1(1))$, $(I(6,1),P(3,8))$, $(I(7,8),R_1^4(2))$ $(I(9,8),R_0^3(1))$, $(I(13,8),R_1^4(1))$

$I(20,8)$: $(I(0,8),R_0^2(2))$, $(I(2,8),R_1^3(3))$, $(I(5,7),P(8,8))$, $(I(5,8),R_\infty^2(1))$, $(I(7,1),P(2,8))$ $(I(8,8),R_1^3(2))$, $(I(10,8),R_0^2(1))$, $(I(14,8),R_1^3(1))$

$I(21,8)$: $(I(1,8),R_0^1(2))$, $(I(3,8),R_1^2(3))$, $(I(6,7),P(7,8))$, $(I(6,8),R_\infty^1(1))$, $(I(8,1),P(1,8))$ $(I(9,8),R_1^2(2))$, $(I(11,8),R_0^1(1))$, $(I(15,8),R_1^2(1))$

$I(22,8)$: $(I(2,8),R_0^3(2))$, $(I(4,8),R_1^1(3))$, $(I(7,7),P(6,8))$, $(I(7,8),R_\infty^2(1))$, $(I(9,1),P(0,8))$ $(I(10,8),R_1^1(2))$, $(I(12,8),R_0^3(1))$, $(I(16,8),R_1^1(1))$

$I(23,8)$: $(I(3,8),R_0^2(2))$, $(I(5,8),R_1^5(3))$, $(I(8,7),P(5,8))$, $(I(8,8),R_\infty^1(1))$, $(I(11,8),R_1^5(2))$ $(I(13,8),R_0^2(1))$, $(I(17,8),R_1^5(1))$

$I(24,8):\big(I(0,8),R_1^4(4)\big),\ \big(I(4,8),R_0^1(2)\big),\ \big(I(6,8),R_1^4(3)\big),\ \big(I(9,7),P(4,8)\big),\ \big(I(9,8),R_\infty^2(1)\big)$
$\big(I(12,8),R_1^4(2)\big),\ \big(I(14,8),R_0^1(1)\big),\ \big(I(18,8),R_1^4(1)\big)$

$I(25,8):\big(I(1,8),R_1^3(4)\big),\ \big(I(5,8),R_0^3(2)\big),\ \big(I(7,8),R_1^3(3)\big),\ \big(I(10,7),P(3,8)\big),\ \big(I(10,8),R_\infty^1(1)\big)$
$\big(I(13,8),R_1^3(2)\big),\ \big(I(15,8),R_0^3(1)\big),\ \big(I(19,8),R_1^3(1)\big)$

$I(26,8):\big(I(2,8),R_1^2(4)\big),\ \big(I(6,8),R_0^2(2)\big),\ \big(I(8,8),R_1^2(3)\big),\ \big(I(11,7),P(2,8)\big),\ \big(I(11,8),R_\infty^2(1)\big)$
$\big(I(14,8),R_1^2(2)\big),\ \big(I(16,8),R_0^2(1)\big),\ \big(I(20,8),R_1^2(1)\big)$

$I(27,8):\big(I(3,8),R_1^1(4)\big),\ \big(I(7,8),R_0^1(2)\big),\ \big(I(9,8),R_1^1(3)\big),\ \big(I(12,7),P(1,8)\big),\ \big(I(12,8),R_\infty^1(1)\big)$
$\big(I(15,8),R_1^1(2)\big),\ \big(I(17,8),R_0^1(1)\big),\ \big(I(21,8),R_1^1(1)\big)$

$I(28,8):\big(I(4,8),R_1^5(4)\big),\ \big(I(8,8),R_0^3(2)\big),\ \big(I(10,8),R_1^5(3)\big),\ \big(I(13,7),P(0,8)\big),\ \big(I(13,8),R_\infty^2(1)\big)$
$\big(I(16,8),R_1^5(2)\big),\ \big(I(18,8),R_0^3(1)\big),\ \big(I(22,8),R_1^5(1)\big)$

$I(29,8):\big(I(5,8),R_1^4(4)\big),\ \big(I(9,8),R_0^2(2)\big),\ \big(I(11,8),R_1^4(3)\big),\ \big(I(14,8),R_\infty^1(1)\big),\ \big(I(17,8),R_1^4(2)\big)$
$\big(I(19,8),R_0^2(1)\big),\ \big(I(23,8),R_1^4(1)\big)$

$I(30,8):\big(I(6,8),R_1^3(4)\big),\ \big(I(10,8),R_0^1(2)\big),\ \big(I(12,8),R_1^3(3)\big),\ \big(I(15,8),R_\infty^2(1)\big),\ \big(I(18,8),R_1^3(2)\big)$
$\big(I(20,8),R_0^1(1)\big),\ \big(I(24,8),R_1^3(1)\big),\ \big(2I(0,8),P(29,8)\big)$

$I(n,8):\big(I(n-24,8),R_1^{(-n+32)\bmod 5+1}(4)\big),\ \big(I(n-20,8),R_0^{(-n+30)\bmod 3+1}(2)\big),\ \big(I(n-18,8),R_1^{(-n+32)\bmod 5+1}(3)\big)$
$\big(I(n-15,8),R_\infty^{(-n+31)\bmod 2+1}(1)\big),\ \big(I(n-12,8),R_1^{(-n+32)\bmod 5+1}(2)\big),\ \big(I(n-10,8),R_0^{(-n+30)\bmod 3+1}(1)\big)$
$\big(I(n-6,8),R_1^{(-n+32)\bmod 5+1}(1)\big),\ \big((v+1)I,vP\big),\ n>30$

Modules of the form $I(n,9)$

Defect: $\partial I(n,9)=3$, for $n\geq 0$.

$I(0,9):-$

$I(1,9):\big(I(0,1),I(6,8)\big),\ \big(I(0,2),P(0,8)\big),\ \big(I(0,4),P(0,1)\big),\ \big(I(0,5),P(2,8)\big),\ \big(I(0,6),R_1^5(1)\big)$
$\big(I(0,7),I(8,8)\big),\ \big(I(0,8),I(3,1)\big)$

$I(2,9):\big(I(0,9),R_1^2(1)\big),\ \big(I(1,1),I(7,8)\big),\ \big(I(1,5),P(1,8)\big),\ \big(I(1,6),R_1^4(1)\big),\ \big(I(1,7),I(9,8)\big)$
$\big(I(1,8),I(4,1)\big)$

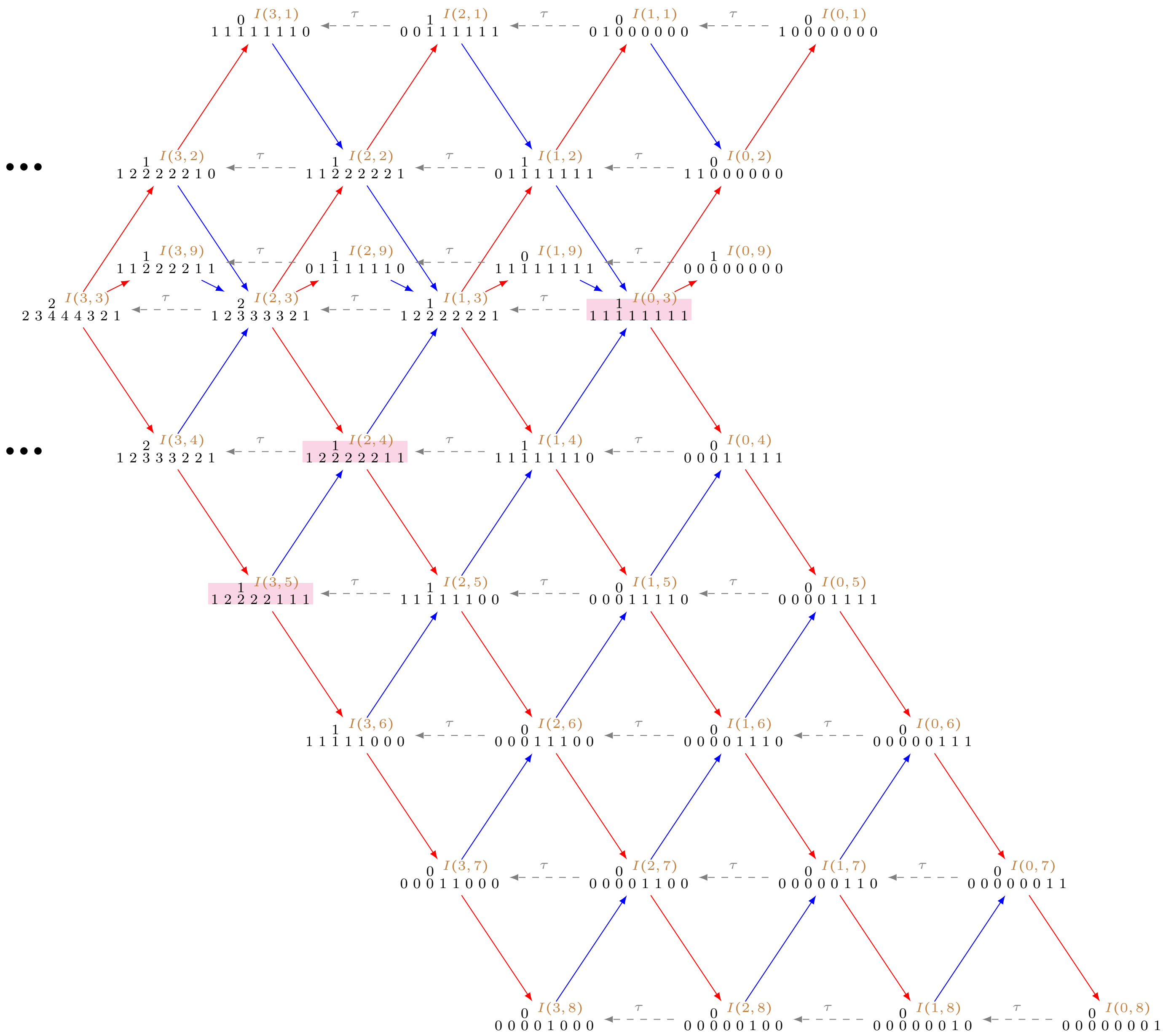
$I(3,9):\big(I(0,1),I(12,8)\big),\ \big(I(0,8),I(7,7)\big),\ \big(I(1,9),R_1^1(1)\big),\ \big(I(2,1),I(8,8)\big),\ \big(I(2,5),P(0,8)\big)$
$\big(I(2,6),R_1^3(1)\big),\ \big(I(2,7),I(10,8)\big),\ \big(I(2,8),I(5,1)\big)$

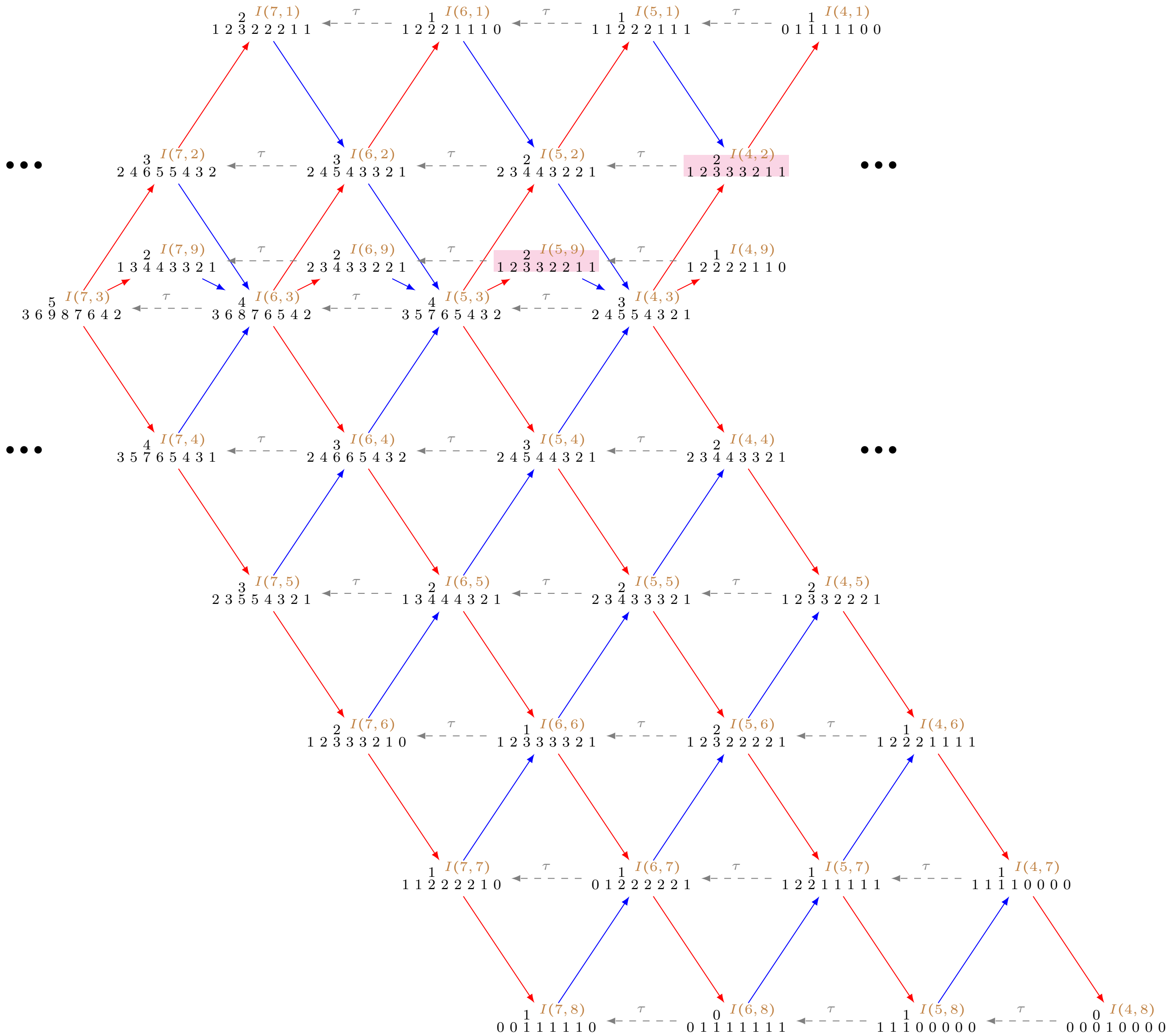
$I(4,9):\big(I(1,1),I(13,8)\big),\ \big(I(1,8),I(8,7)\big),\ \big(I(2,9),R_1^5(1)\big),\ \big(I(3,1),I(9,8)\big),\ \big(I(3,6),R_1^2(1)\big)$
$\big(I(3,7),I(11,8)\big),\ \big(I(3,8),I(6,1)\big)$

$I(5,9):\big(I(0,8),I(9,1)\big),\ \big(I(2,1),I(14,8)\big),\ \big(I(2,8),I(9,7)\big),\ \big(I(3,9),R_1^4(1)\big),\ \big(I(4,1),I(10,8)\big)$
$\big(I(4,6),R_1^1(1)\big),\ \big(I(4,7),I(12,8)\big),\ \big(I(4,8),I(7,1)\big)$

$I(n,9):\big(I(n-5,8),I(n+4,1)\big),\ \big(I(n-3,1),I(n+9,8)\big),\ \big(I(n-3,8),I(n+4,7)\big)$
$\big(I(n-2,9),R_1^{(-n+8)\bmod 5+1}(1)\big),\ \big(I(n-1,1),I(n+5,8)\big),\ \big(I(n-1,6),R_1^{(-n+5)\bmod 5+1}(1)\big)$

$$\big(I(n-1,7), I(n+7,8)\big),\ \big(I(n-1,8), I(n+2,1)\big),\ n > 5$$

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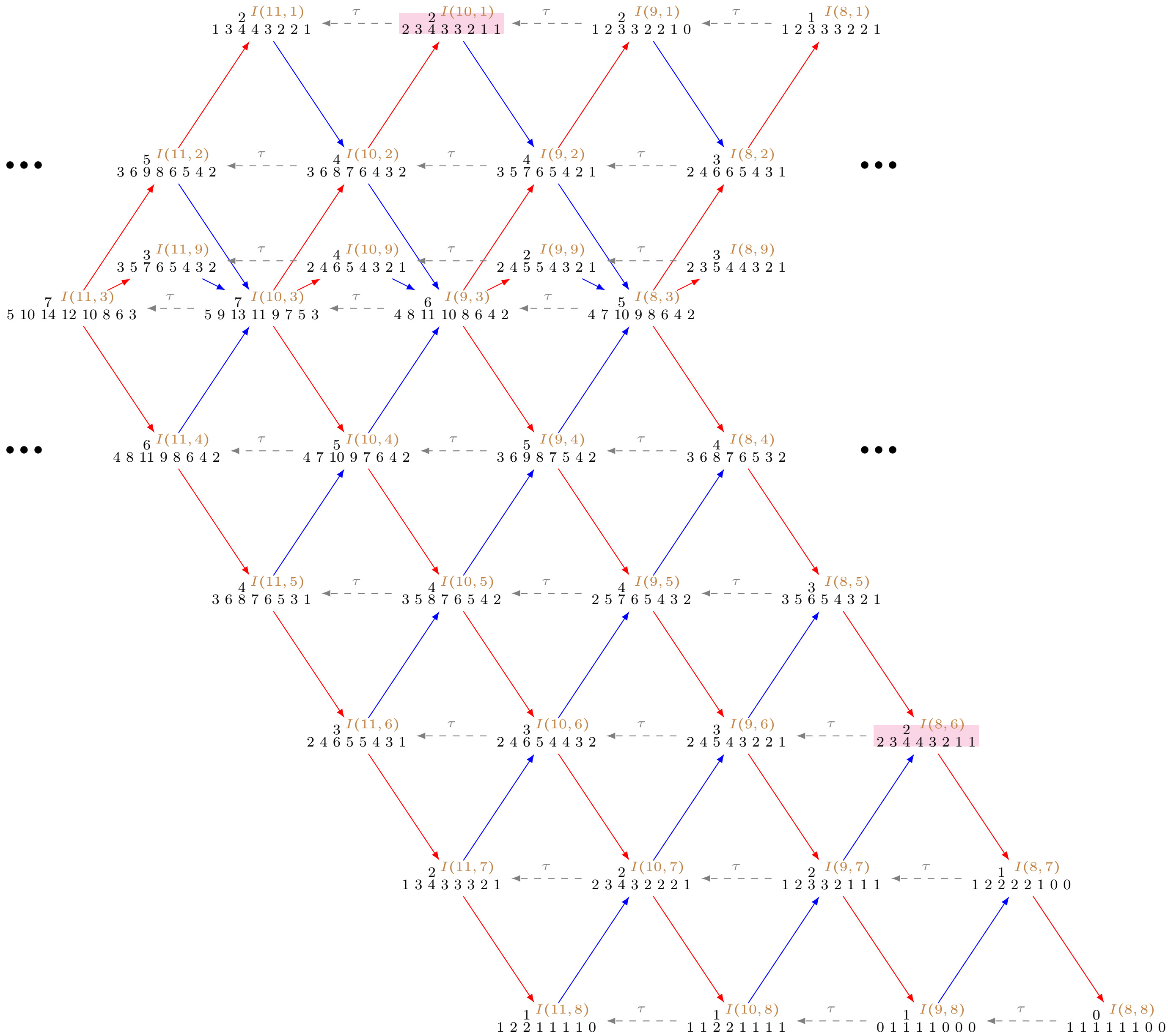
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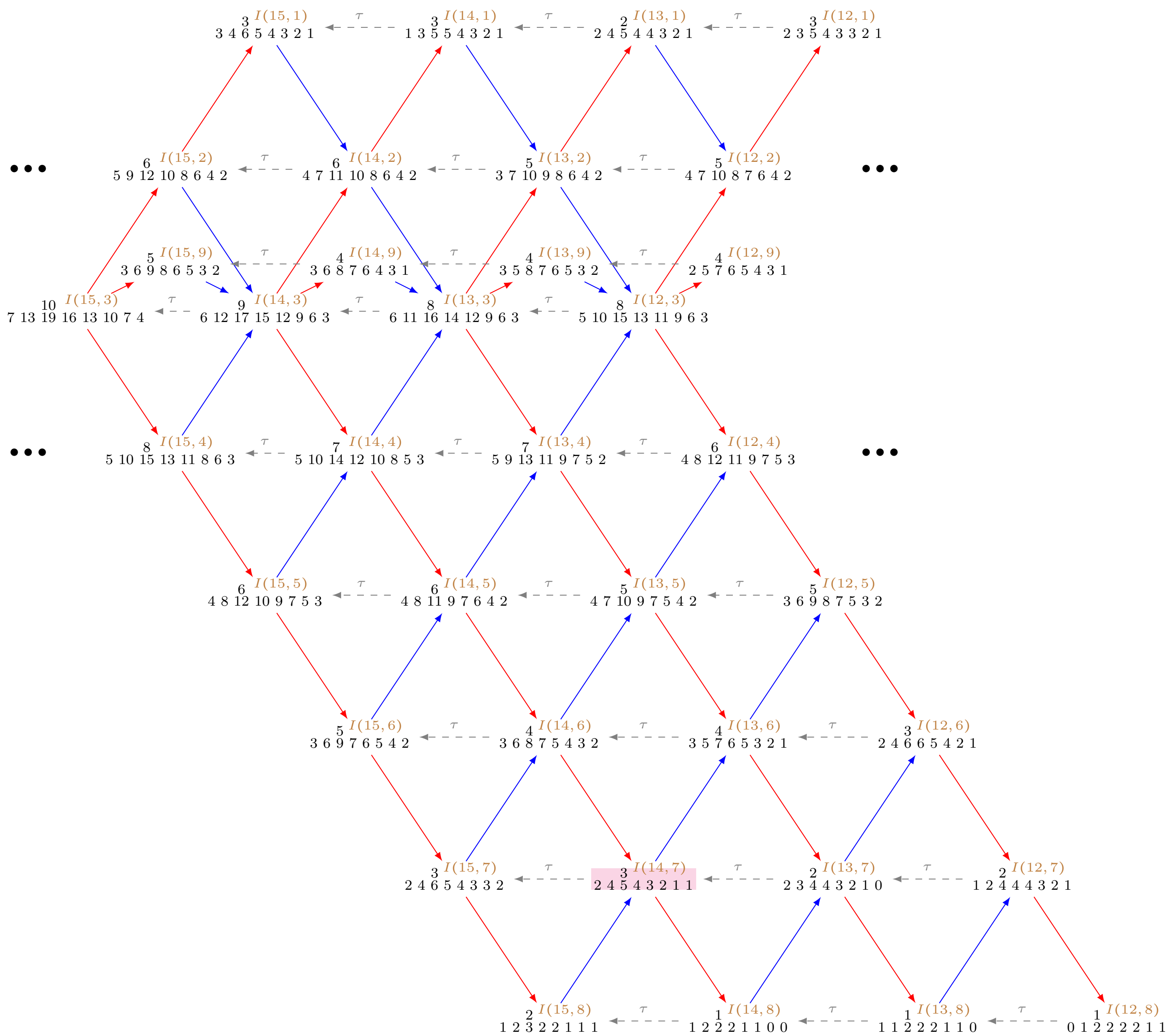

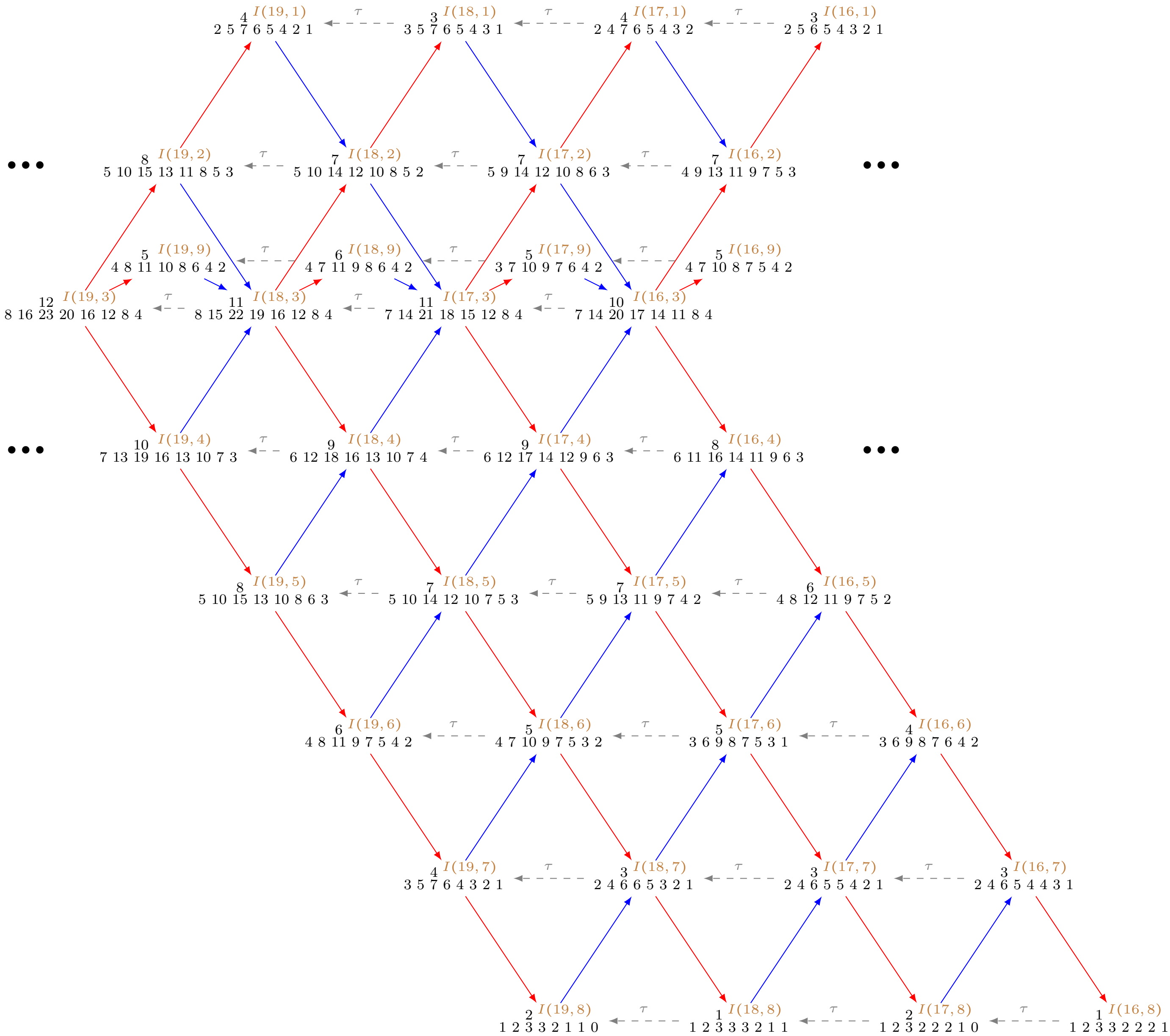

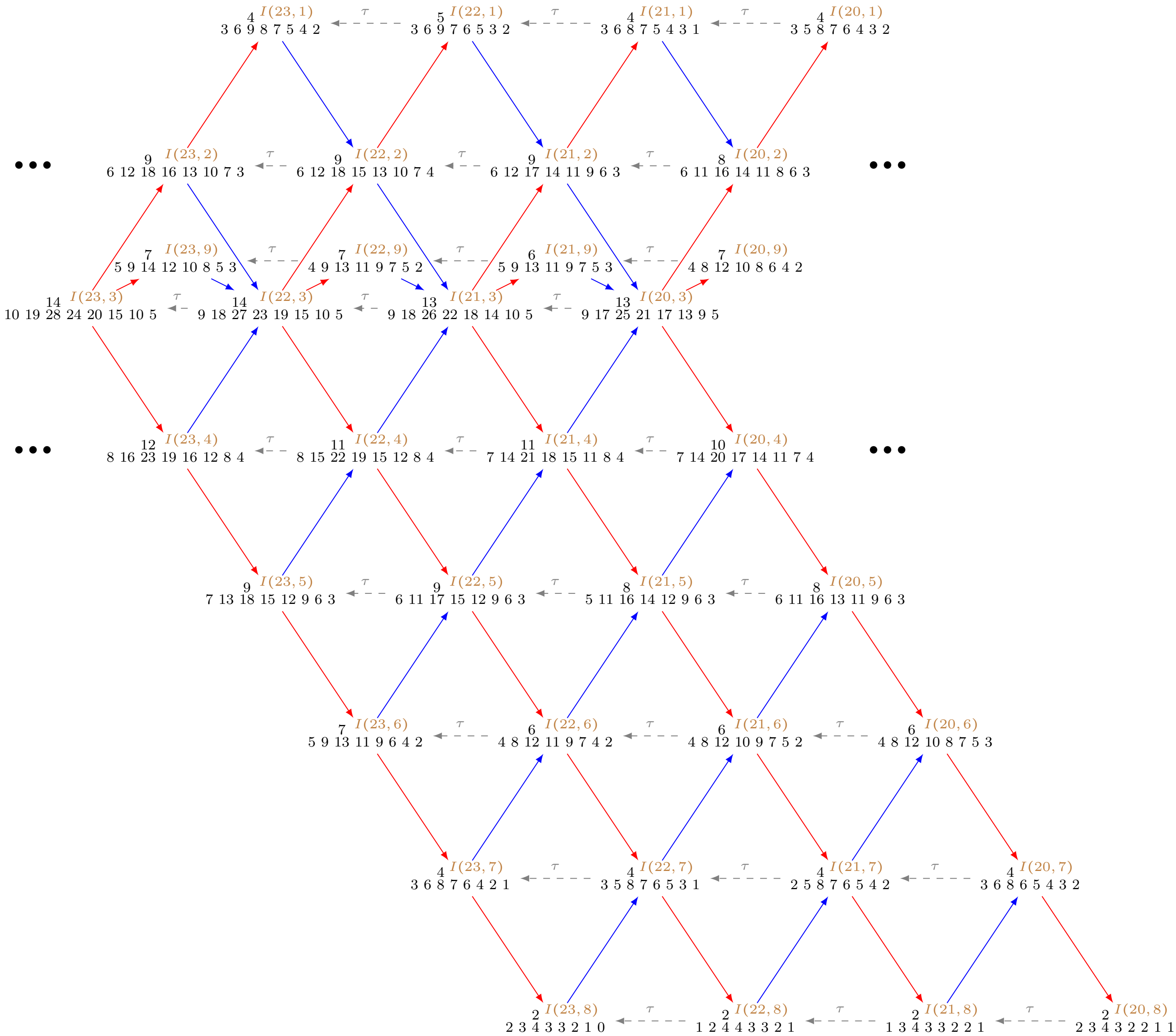
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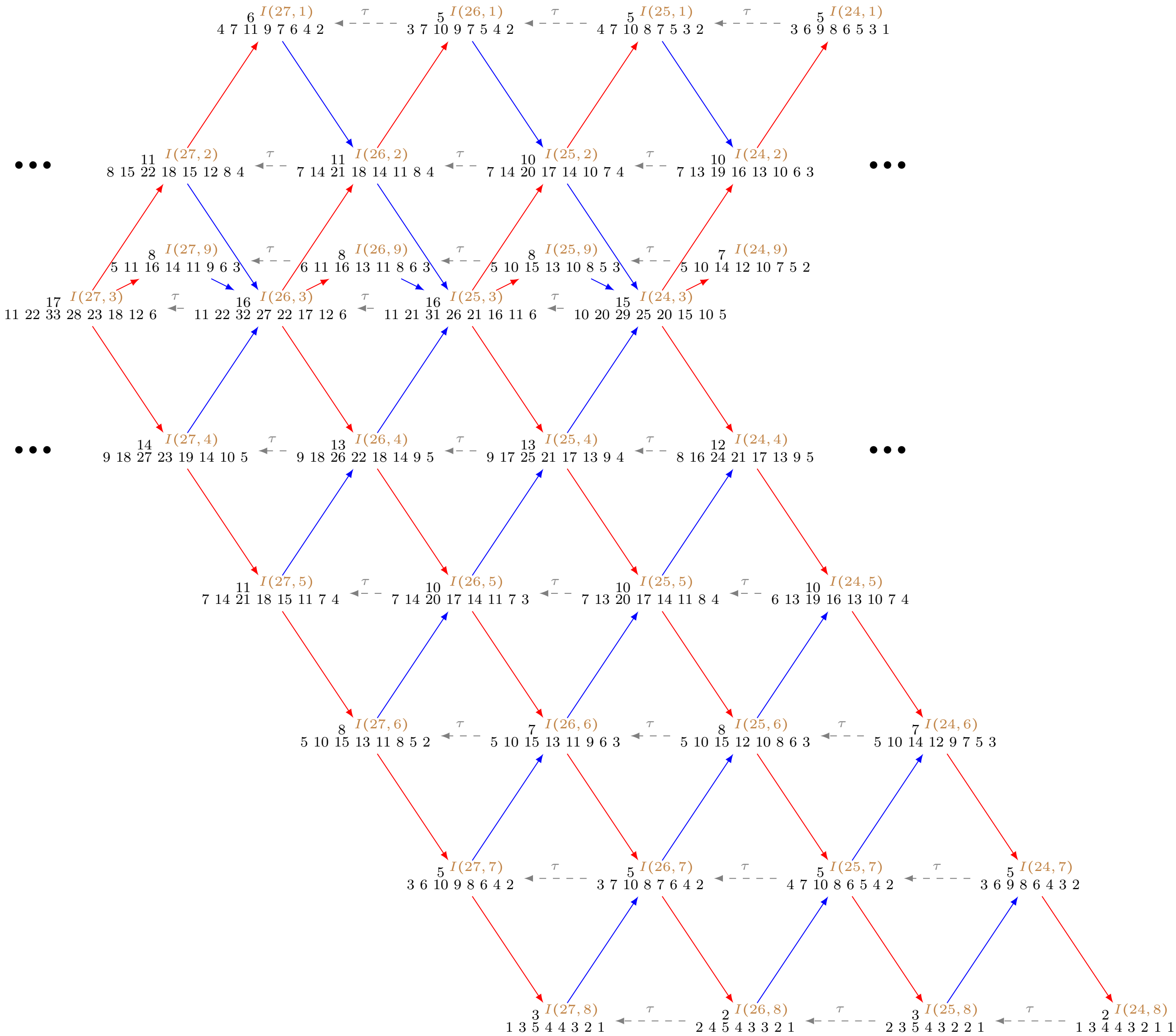

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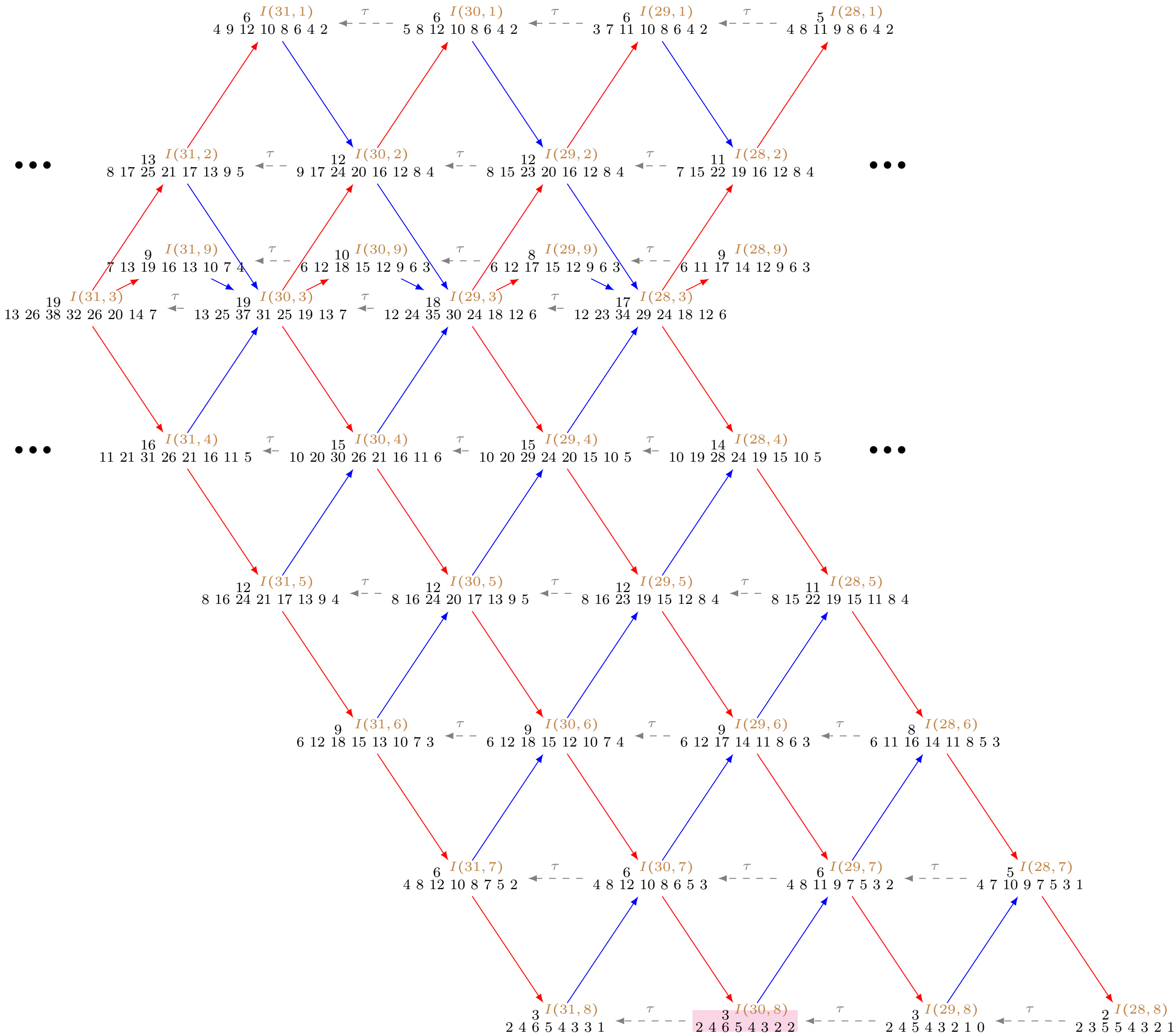
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I(29,8)
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Schofield pairs associated to regular exceptional modules

The non-homogeneous tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{E}}_8)}$

$R_1^1(1) : \big(I(0,9), P(0,6)\big),\ \big(I(2,6), P(0,9)\big),\ \big(I(2,7), P(1,1)\big),\ \big(I(2,8), P(3,8)\big)$

$R_1^1(2) : \big(R_1^2(1), R_1^1(1)\big),\ \big(I(4,1), P(0,7)\big),\ \big(I(2,6), P(1,6)\big),\ \big(I(2,7), P(4,1)\big),\ \big(I(7,8), P(4,8)\big)$
$\big(I(2,8), P(9,8)\big)$

$R_1^2(1) : \big(I(1,5), P(0,2)\big),\ \big(I(1,1), P(0,7)\big),\ \big(I(1,6), P(1,9)\big),\ \big(I(1,7), P(2,1)\big),\ \big(I(1,8), P(4,8)\big)$

$R_1^2(2) : \big(R_1^3(1), R_1^2(1)\big),\ \big(I(6,7), P(0,1)\big),\ \big(I(11,8), P(0,8)\big),\ \big(I(3,1), P(1,7)\big),\ \big(I(1,6), P(2,6)\big)$
$\big(I(1,7), P(5,1)\big),\ \big(I(6,8), P(5,8)\big),\ \big(I(1,8), P(10,8)\big)$

$R_1^3(1) : \big(I(2,1), P(0,1)\big),\ \big(I(5,8), P(0,8)\big),\ \big(I(1,9), P(0,9)\big),\ \big(I(0,5), P(1,2)\big),\ \big(I(0,1), P(1,7)\big)$
$\big(I(0,6), P(2,9)\big),\ \big(I(0,7), P(3,1)\big),\ \big(I(0,8), P(5,8)\big)$

$R_1^3(2) : \big(R_1^4(1), R_1^3(1)\big),\ \big(I(5,7), P(1,1)\big),\ \big(I(10,8), P(1,8)\big),\ \big(I(2,1), P(2,7)\big),\ \big(I(0,6), P(3,6)\big)$
$\big(I(0,7), P(6,1)\big),\ \big(I(5,8), P(6,8)\big),\ \big(I(0,8), P(11,8)\big)$

$R_1^4(1) : \big(I(1,1), P(1,1)\big),\ \big(I(4,8), P(1,8)\big),\ \big(I(0,9), P(1,9)\big)$

$R_1^4(2) : \big(R_1^5(1), R_1^4(1)\big),\ \big(I(4,7), P(2,1)\big),\ \big(I(9,8), P(2,8)\big),\ \big(I(1,1), P(3,7)\big),\ \big(I(4,8), P(7,8)\big)$

$R_1^5(1) : \big(I(3,7), P(0,1)\big),\ \big(I(0,2), P(0,5)\big),\ \big(I(0,1), P(2,1)\big),\ \big(I(3,8), P(2,8)\big)$

$R_1^5(2) : \big(R_1^1(1), R_1^5(1)\big),\ \big(I(3,6), P(0,6)\big),\ \big(I(3,7), P(3,1)\big),\ \big(I(8,8), P(3,8)\big),\ \big(I(0,1), P(4,7)\big)$
$\big(I(3,8), P(8,8)\big)$

$R_1^5(3) : \big(R_1^1(2), R_1^5(1)\big),\ \big(R_1^2(1), R_1^5(2)\big),\ \big(I(8,7), P(0,7)\big),\ \big(I(13,8), P(4,8)\big),\ \big(I(3,7), P(5,7)\big)$
$\big(I(8,8), P(9,8)\big),\ \big(I(3,8), P(14,8)\big)$

$R_1^5(4) : \big(R_1^1(3), R_1^5(1)\big),\ \big(R_1^2(2), R_1^5(2)\big),\ \big(R_1^3(1), R_1^5(3)\big),\ \big(I(23,8), P(0,8)\big),\ \big(I(18,8), P(5,8)\big)$
$\big(I(13,8), P(10,8)\big),\ \big(I(8,8), P(15,8)\big),\ \big(I(3,8), P(20,8)\big)$

$R_1^1(3) : \big(R_1^2(2), R_1^1(1)\big),\ \big(R_1^3(1), R_1^1(2)\big),\ \big(I(17,8), P(0,8)\big),\ \big(I(7,7), P(1,7)\big),\ \big(I(12,8), P(5,8)\big)$
$\big(I(2,7), P(6,7)\big),\ \big(I(7,8), P(10,8)\big),\ \big(I(2,8), P(15,8)\big)$

$R_1^1(4) : \big(R_1^2(3), R_1^1(1)\big),\ \big(R_1^3(2), R_1^1(2)\big),\ \big(R_1^4(1), R_1^1(3)\big),\ \big(I(22,8), P(1,8)\big),\ \big(I(17,8), P(6,8)\big)$
$\big(I(12,8), P(11,8)\big),\ \big(I(7,8), P(16,8)\big),\ \big(I(2,8), P(21,8)\big)$

$R_1^2(3) : \big(R_1^3(2), R_1^2(1)\big),\ \big(R_1^4(1), R_1^2(2)\big),\ \big(I(16,8), P(1,8)\big),\ \big(I(6,7), P(2,7)\big),\ \big(I(11,8), P(6,8)\big)$
$\big(I(1,7), P(7,7)\big),\ \big(I(6,8), P(11,8)\big),\ \big(I(1,8), P(16,8)\big)$

$R_1^2(4) : \big(R_1^3(3), R_1^2(1)\big),\ \big(R_1^4(2), R_1^2(2)\big),\ \big(R_1^5(1), R_1^2(3)\big),\ \big(I(21,8), P(2,8)\big),\ \big(I(16,8), P(7,8)\big)$
$\big(I(11,8), P(12,8)\big),\ \big(I(6,8), P(17,8)\big),\ \big(I(1,8), P(22,8)\big)$

$R_1^3(3) : \big(R_1^4(2), R_1^3(1)\big),\ \big(R_1^5(1), R_1^3(2)\big),\ \big(I(15,8), P(2,8)\big),\ \big(I(5,7), P(3,7)\big),\ \big(I(10,8), P(7,8)\big)$
$\big(I(0,7), P(8,7)\big),\ \big(I(5,8), P(12,8)\big),\ \big(I(0,8), P(17,8)\big)$

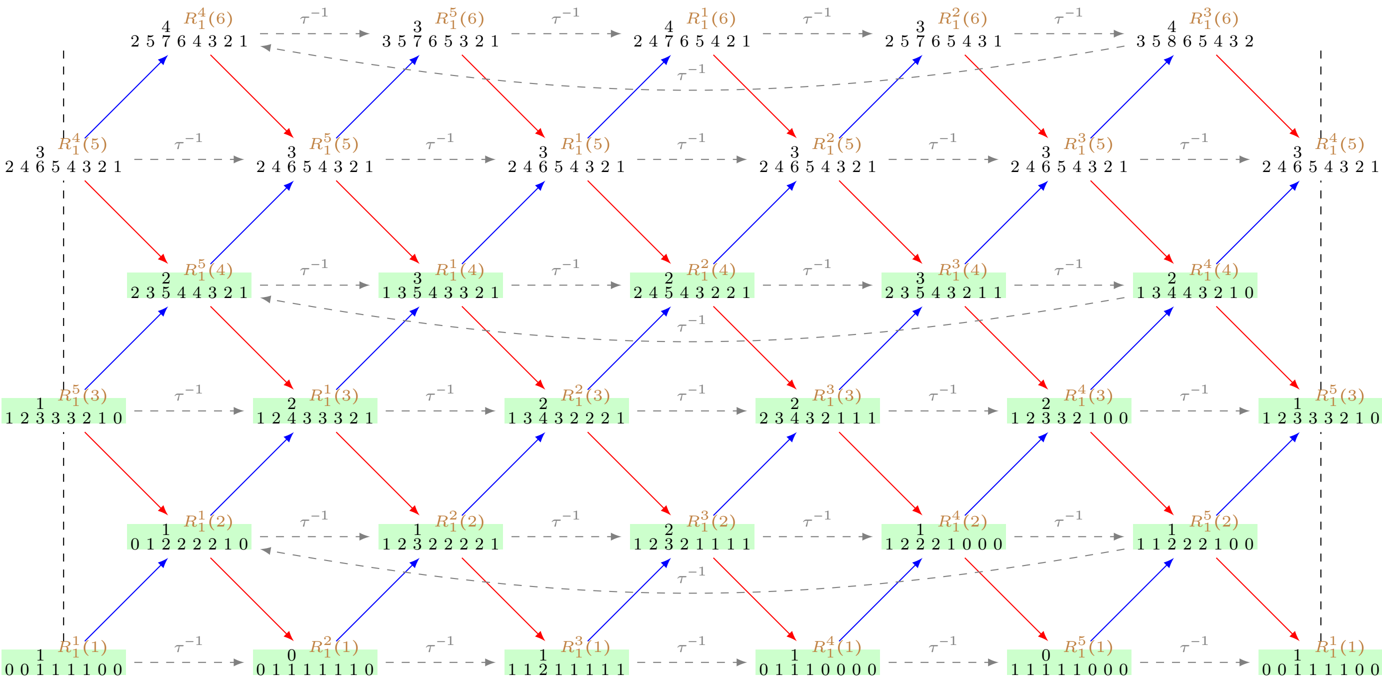
$R_1^3(4) : \big(R_1^4(3), R_1^3(1)\big),\ \big(R_1^5(2), R_1^3(2)\big),\ \big(R_1^1(1), R_1^3(3)\big),\ \big(I(20,8), P(3,8)\big),\ \big(I(15,8), P(8,8)\big)$
$\big(I(10,8), P(13,8)\big),\ \big(I(5,8), P(18,8)\big),\ \big(I(0,8), P(23,8)\big)$

$R_1^4(3) : \big(R_1^5(2), R_1^4(1)\big),\ \big(R_1^1(1), R_1^4(2)\big),\ \big(I(14,8), P(3,8)\big),\ \big(I(4,7), P(4,7)\big),\ \big(I(9,8), P(8,8)\big)$

$\big(I(4,8), P(13,8)\big)$

$R_1^4(4) : \big(R_1^5(3), R_1^4(1)\big),\ \big(R_1^1(2), R_1^4(2)\big),\ \big(R_1^2(1), R_1^4(3)\big),\ \big(I(19,8), P(4,8)\big),\ \big(I(14,8), P(9,8)\big)$
$\big(I(9,8), P(14,8)\big),\ \big(I(4,8), P(19,8)\big)$

$\mathcal{T}_1^{\Delta(\widetilde{\mathbb{E}}_8)}$



The non-homogeneous tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{E}}_8)}$

$R_0^1(1) : \big(I(9,8), P(0,8)\big),\ \big(I(3,7), P(1,7)\big),\ \big(I(2,1), P(2,1)\big),\ \big(I(6,8), P(3,8)\big),\ \big(I(0,7), P(4,7)\big)$
$\big(I(3,8), P(6,8)\big),\ \big(I(0,8), P(9,8)\big)$

$R_0^1(2) : \big(R_0^2(1), R_0^1(1)\big),\ \big(I(18,8), P(1,8)\big),\ \big(I(15,8), P(4,8)\big),\ \big(I(12,8), P(7,8)\big),\ \big(I(9,8), P(10,8)\big)$
$\big(I(6,8), P(13,8)\big),\ \big(I(3,8), P(16,8)\big),\ \big(I(0,8), P(19,8)\big)$

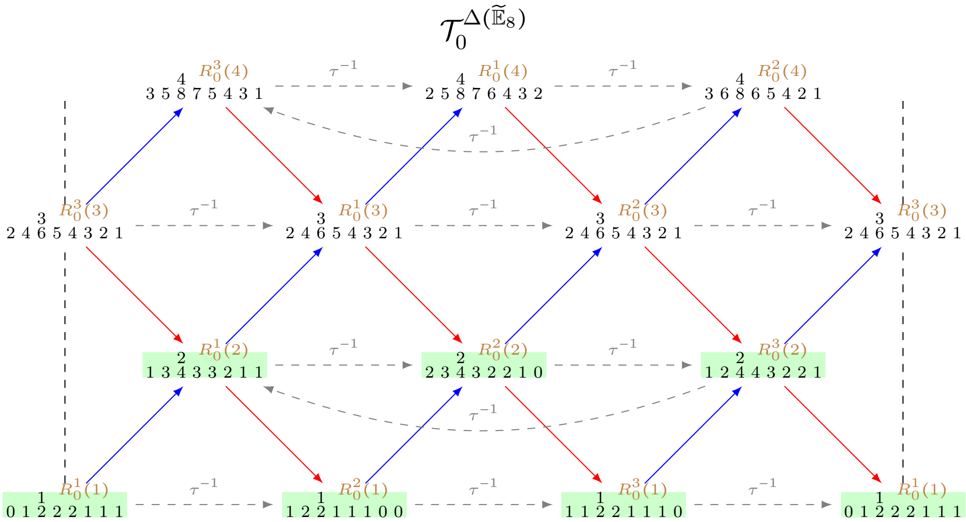
$R_0^2(1) : \big(I(4,1), P(0,1)\big),\ \big(I(8,8), P(1,8)\big),\ \big(I(2,7), P(2,7)\big),\ \big(I(1,1), P(3,1)\big),\ \big(I(5,8), P(4,8)\big)$
$\big(I(2,8), P(7,8)\big)$

$R_0^2(2) : \big(R_0^3(1), R_0^2(1)\big),\ \big(I(17,8), P(2,8)\big),\ \big(I(14,8), P(5,8)\big),\ \big(I(11,8), P(8,8)\big),\ \big(I(8,8), P(11,8)\big)$
$\big(I(5,8), P(14,8)\big),\ \big(I(2,8), P(17,8)\big)$

$R_0^3(1) : \big(I(4,7), P(0,7)\big),\ \big(I(3,1), P(1,1)\big),\ \big(I(7,8), P(2,8)\big),\ \big(I(1,7), P(3,7)\big),\ \big(I(0,1), P(4,1)\big)$
$\big(I(4,8), P(5,8)\big),\ \big(I(1,8), P(8,8)\big)$

$R_0^3(2) : \big(R_0^1(1), R_0^3(1)\big),\ \big(I(19,8), P(0,8)\big),\ \big(I(16,8), P(3,8)\big),\ \big(I(13,8), P(6,8)\big),\ \big(I(10,8), P(9,8)\big)$

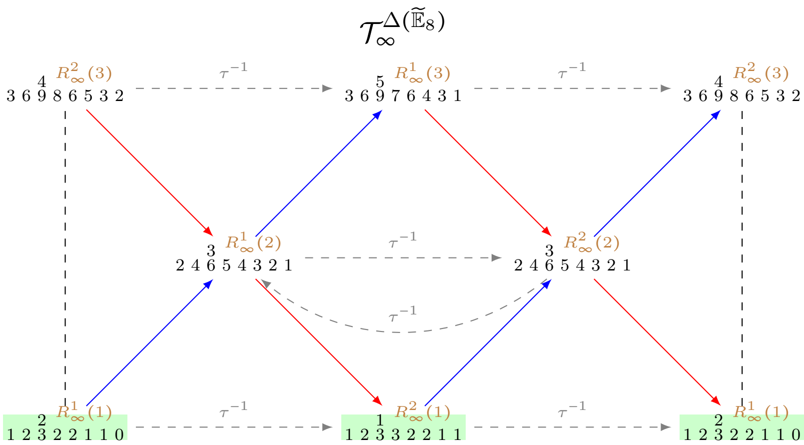
$\big(I(7,8),P(12,8)\big),\ \big(I(4,8),P(15,8)\big),\ \big(I(1,8),P(18,8)\big)$



The non-homogeneous tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{E}}_8)}$

$R^1_\infty(1):\big(I(13,8),P(1,8)\big),\ \big(I(11,8),P(3,8)\big),\ \big(I(9,8),P(5,8)\big),\ \big(I(7,8),P(7,8)\big),\ \big(I(5,8),P(9,8)\big)$
$\big(I(3,8),P(11,8)\big),\ \big(I(1,8),P(13,8)\big)$

$R^2_\infty(1):\big(I(14,8),P(0,8)\big),\ \big(I(12,8),P(2,8)\big),\ \big(I(10,8),P(4,8)\big),\ \big(I(8,8),P(6,8)\big),\ \big(I(6,8),P(8,8)\big)$
$\big(I(4,8),P(10,8)\big),\ \big(I(2,8),P(12,8)\big),\ \big(I(0,8),P(14,8)\big)$



Appendix B

Preprojective indecomposables of defect -2 and dimension up to 4δ

Consider the quiver orientations and labelings presented in Theorem 2.6. Let P_0 be the projective simple of defect -1 corresponding to the unique sink 1.

We list all the preprojective indecomposable modules P and a corresponding orthogonal exceptional pair (P'', P') such that:

(1) $\partial P = -2$;

(2) $\underline{\dim} P < 4\delta$;

(3) $\underline{\dim} P - \underline{\dim} P_0$ is a root, thus it is the dimension of a unique preprojective P_1 of defect -1.

We also list $n = \langle \underline{\dim} P_0, \underline{\dim} P \rangle$ and the Ringel-Hall numbers $F^P_{P_1 P_0} = S_g(P_1, P', P'', P_0) - S_g(P_1, P'', P'', P_0)$. In case $P' = P_0$ we know that $F^P_{P_1 P_0} = 1$.

The information is given in units of the form:

$\underline{\dim} P$,	$\underline{\dim} P'$,	$\underline{\dim} P''$,
n,	$F^P_{P_1 P_0}$.	

Here $F^P_{P_1 P_0} = S_g(P_1, P', P'', P_0) - S_g(P_1, P'', P'', P_0)$, $P(t,i) = \tau^{-t} P_i$ and when $P = P(t,i)$ the corresponding orthogonal exceptional pair will be denoted by $(P''(t,i), P'(t,i))$.

B.1 Type $\widetilde{\mathbb{D}}_m$

$\underline{\dim} P < 2\delta$

$\underline{\dim} P(1, m+1) = \left(\begin{smallmatrix} 2 & & & & & & 1 \\ & 3 & 2 & 2 & \cdots & 2 & \\ 1 & & & & & & 1 \end{smallmatrix}\right)$, $\underline{\dim} P'(1, m+1) = \left(\begin{smallmatrix} 0 & & & & & & 0 \\ & 1 & 0 & 0 & \cdots & 0 & \\ 0 & & & & & & 0 \end{smallmatrix}\right)$, $\underline{\dim} P''(1, m+1) = \left(\begin{smallmatrix} 2 & & & & & & 1 \\ & 2 & 2 & 2 & \cdots & 2 & \\ 1 & & & & & & 1 \end{smallmatrix}\right)$,
$n = 2$, $F^P_{P_1 P_0} = {}^0 a_1(q) - 0 = q - 2 = f_1(q)$.

$\underline{\dim} P(2, m) = \left(\begin{smallmatrix} 2 & & & & & & 1 \\ & 3 & 3 & 2 & \cdots & 2 & \\ 1 & & & & & & 1 \end{smallmatrix}\right)$, $\underline{\dim} P'(2, m) = \left(\begin{smallmatrix} 1 & & & & & & 0 \\ & 1 & 1 & 0 & \cdots & 0 & \\ 1 & & & & & & 0 \end{smallmatrix}\right)$, $\underline{\dim} P''(2, m) = \left(\begin{smallmatrix} 1 & & & & & & 1 \\ & 2 & 2 & 2 & \cdots & 2 & \\ 0 & & & & & & 1 \end{smallmatrix}\right)$,
$n = 2$, $F^P_{P_1 P_0} = {}^2 a_0(q) - {}^1 a_0(q) = q - 2 = f_1(q)$.

$\underline{\dim} P(3, m-1) = \left({}^{2}_{1}\, 3\,3\,3\,2 \cdots 2\, {}^{1}_{1}\right)$, $\underline{\dim} P'(3, m-1) = \left({}^{0}_{0}\, 1\,1\,1\,0 \cdots 0\, {}^{0}_{0}\right)$, $\underline{\dim} P''(3, m-1) = \left({}^{2}_{1}\, 2\,2\,2\,2 \cdots 2\, {}^{1}_{1}\right)$,
$n = 2$, $F^{P}_{P_1 P_0} = {}^{0}a_1(q) - 0 = q - 2 = f_1(q)$.

$\underline{\dim} P(4, m-2) = \left({}^{2}_{1}\, 3\,3\,3\,3\,2 \cdots 2\, {}^{1}_{1}\right)$, $\underline{\dim} P'(4, m-2) = \left({}^{1}_{1}\, 1\,1\,1\,1\,0 \cdots 0\, {}^{0}_{0}\right)$, $\underline{\dim} P''(4, m-2) = \left({}^{1}_{0}\, 2\,2\,2\,2\,2 \cdots 2\, {}^{1}_{1}\right)$,
$n = 2$, $F^{P}_{P_1 P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q - 2 = f_1(q)$.

$\vdots$

- In case m is even:

$\underline{\dim} P(m-3, 5) = \left({}^{2}_{1}\, 3 \cdots 3\, {}^{1}_{1}\right)$, $\underline{\dim} P'(m-3, 5) = \left({}^{0}_{0}\, 1 \cdots 1\, {}^{0}_{0}\right)$, $\underline{\dim} P''(m-3, 5) = \left({}^{2}_{1}\, 2 \cdots 2\, {}^{1}_{1}\right)$,
$n = 2$, $F^{P}_{P_1 P_0} = {}^{0}a_1(q) - 0 = q - 2 = f_1(q)$.

- In case m is odd:

$\underline{\dim} P(m-3, 5) = \left({}^{2}_{1}\, 3 \cdots 3\, {}^{1}_{1}\right)$, $\underline{\dim} P'(m-3, 5) = \left({}^{1}_{1}\, 1 \cdots 1\, {}^{0}_{0}\right)$, $\underline{\dim} P''(m-3, 5) = \left({}^{1}_{0}\, 2 \cdots 2\, {}^{1}_{1}\right)$,
$n = 2$, $F^{P}_{P_1 P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q - 2 = f_1(q)$.

- For $i \in \{5, \ldots, m+1\}$, having $i - 4$ ones on the central axis of the dimensions:

$\underline{\dim} P(0, i) = \left({}^{1}_{0}\, 1 \cdots 1\,0 \cdots 0\, {}^{0}_{0}\right)$, $\underline{\dim} P'(0, i) = \left({}^{1}_{0}\, 0\,0\,0 \cdots 0\, {}^{0}_{0}\right)$, $\underline{\dim} P''(0, i) = \left({}^{0}_{0}\, 1 \cdots 1\,0 \cdots 0\, {}^{0}_{0}\right)$,
$n = 1$, $F^{P}_{P_1 P_0} = 1$.

$2\delta < \underline{\dim} P < 4\delta$

$\underline{\dim} P(m-1, m+1) = 2\delta + \underline{\dim} P(1, m+1)$, $\underline{\dim} P'(m-1, m+1) = \delta + \underline{\dim} P'(1, m+1)$,
$\underline{\dim} P''(m-1, m+1) = \delta + \underline{\dim} P''(1, m+1)$,
$n = 4$, $F^{P}_{P_1 P_0} = {}^{0}a_2(q) - {}^{3}a_0(q) = f_3(q)$.

$\underline{\dim} P(m, m) = 2\delta + \underline{\dim} P(2, m)$, $\underline{\dim} P'(m, m) = \delta + \underline{\dim} P'(2, m)$,
$\underline{\dim} P''(m, m) = \delta + \underline{\dim} P''(2, m)$,
$n = 4$, $F^{P}_{P_1 P_0} = {}^{2}a_1(q) - {}^{1}a_1(q) = f_3(q)$.

$\underline{\dim} P(m+1, m-1) = 2\delta + \underline{\dim} P(3, m-1)$, $\underline{\dim} P'(m+1, m-1) = \delta + \underline{\dim} P'(3, m-1)$,
$\underline{\dim} P''(m+1, m-1) = \delta + \underline{\dim} P''(3, m-1)$,
$n = 4$, $F^{P}_{P_1 P_0} = {}^{0}a_2(q) - {}^{3}a_0(q) = f_3(q)$.

$\vdots$

- In case m is even:

$\underline{\dim} P(2m-5,5) = 2\delta + \underline{\dim} P(m-3,5)$, $\underline{\dim} P'(2m-5,5) = \delta + \underline{\dim} P'(m-3,5)$,
$\underline{\dim} P''(2m-5,5) = \delta + \underline{\dim} P''(m-3,5)$,
$n = 4$, $F^{P}_{P_1 P_0} = {}^{0}a_2(q) - {}^{3}a_0(q) = f_3(q)$.

- In case m is odd:

$\underline{\dim} P(2m-5,5) = 2\delta + \underline{\dim} P(m-3,5)$, $\underline{\dim} P'(2m-5,5) = \delta + \underline{\dim} P'(m-3,5)$,
$\underline{\dim} P''(2m-5,5) = \delta + \underline{\dim} P''(m-3,5)$,
$n = 4$, $F^{P}_{P_1 P_0} = {}^{2}a_1(q) - {}^{1}a_1(q) = f_3(q)$.

- For $i \in \{5, \dots, m+1\}$:

$\underline{\dim} P(m-2,i) = 2\delta + \underline{\dim} P(0,i)$, $\underline{\dim} P'(m-2,i) = \delta + \underline{\dim} P'(0,i)$,
$\underline{\dim} P''(m-2,1) = \delta + \underline{\dim} P''(0,i)$,
$n = 3$, $F^{P}_{P_1 P_0} = {}^{3}a_0(q) - {}^{0}a_1(q) = f_2(q)$.

B.2 Type $\widetilde{\mathbb{E}}_6$

$\underline{\dim} P < 2\delta$

$\underline{\dim} P(0,2) = \left(\begin{smallmatrix} &&0&& \\ &&0&& \\ 1&1&0&0&0 \end{smallmatrix}\right)$, $\underline{\dim} P'(0,2) = \left(\begin{smallmatrix} &&0&& \\ &&0&& \\ 1&0&0&0&0 \end{smallmatrix}\right)$, $\underline{\dim} P''(0,2) = \left(\begin{smallmatrix} &&0&& \\ &&0&& \\ 0&1&0&0&0 \end{smallmatrix}\right)$,
$n = 1$, $F^{P}_{P_1 P_0} = 1 = f_0(q)$.

$\underline{\dim} P(0,4) = \left(\begin{smallmatrix} &&0&& \\ &&1&& \\ 1&1&1&0&0 \end{smallmatrix}\right)$, $\underline{\dim} P'(0,4) = \left(\begin{smallmatrix} &&0&& \\ &&0&& \\ 1&0&0&0&0 \end{smallmatrix}\right)$, $\underline{\dim} P''(0,4) = \left(\begin{smallmatrix} &&0&& \\ &&1&& \\ 0&1&1&0&0 \end{smallmatrix}\right)$,
$n = 1$, $F^{P}_{P_1 P_0} = 1 = f_0(q)$.

$\underline{\dim} P(0,6) = \left(\begin{smallmatrix} &&0&& \\ &&0&& \\ 1&1&1&1&0 \end{smallmatrix}\right)$, $\underline{\dim} P'(0,6) = \left(\begin{smallmatrix} &&0&& \\ &&0&& \\ 1&0&0&0&0 \end{smallmatrix}\right)$, $\underline{\dim} P''(0,6) = \left(\begin{smallmatrix} &&0&& \\ &&0&& \\ 0&1&1&1&0 \end{smallmatrix}\right)$,
$n = 1$, $F^{P}_{P_1 P_0} = 1 = f_0(q)$.

$\underline{\dim} P(2,2) = \left(\begin{smallmatrix} &&0&& \\ &&1&& \\ 1&1&2&1&0 \end{smallmatrix}\right)$, $\underline{\dim} P'(2,2) = \left(\begin{smallmatrix} &&0&& \\ &&0&& \\ 0&0&1&0&0 \end{smallmatrix}\right)$, $\underline{\dim} P''(2,2) = \left(\begin{smallmatrix} &&0&& \\ &&1&& \\ 1&1&1&1&0 \end{smallmatrix}\right)$,
$n = 1$, $F^{P}_{P_1 P_0} = {}^{1}a_0(q) - 0 = 1 = 1 = f_0(q)$.

$\underline{\dim} P(3,2) = \delta + \underline{\dim} P(0,2)$, $\underline{\dim} P'(3,2) = \left(\begin{smallmatrix} &&0&& \\ &&1&& \\ 1&1&1&1&0 \end{smallmatrix}\right)$, $\underline{\dim} P''(3,2) = \left(\begin{smallmatrix} &&1&& \\ &&1&& \\ 1&2&2&1&1 \end{smallmatrix}\right)$,
$n = 2$, $F^{P}_{P_1 P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q-1-1 = q-2 = f_1(q)$.

$\underline{\dim} P(3,4) = \delta + \underline{\dim} P(0,4)$, $\underline{\dim} P'(3,4) = \left(\begin{smallmatrix} &&0&& \\ &&1&& \\ 1&1&1&1&0 \end{smallmatrix}\right)$, $\underline{\dim} P''(3,4) = \left(\begin{smallmatrix} &&1&& \\ &&2&& \\ 1&2&3&1&1 \end{smallmatrix}\right)$,
$n = 2$, $F^{P}_{P_1 P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q-1-1 = q-2 = f_1(q)$.

$\underline{\dim} P(3,6) = \delta + \underline{\dim} P(0,6)$, $\underline{\dim} P'(3,6) = \left(\begin{smallmatrix} &&0&& \\ &&1&& \\ 1&1&1&1&0 \end{smallmatrix}\right)$, $\underline{\dim} P''(3,6) = \left(\begin{smallmatrix} &&1&& \\ &&1&& \\ 1&2&3&2&1 \end{smallmatrix}\right)$,
$n = 2$, $F^{P}_{P_1 P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q-1-1 = q-2 = f_1(q)$.

$\underline{\dim} P(5,2) = \delta + \underline{\dim} P(2,2)$, $\underline{\dim} P'(5,2) = \left(\begin{smallmatrix} &&0&& \\ &&1&& \\ 0&1&2&1&0 \end{smallmatrix}\right)$, $\underline{\dim} P''(5,2) = \left(\begin{smallmatrix} &&1&& \\ &&2&& \\ 2&2&3&2&1 \end{smallmatrix}\right)$,

$n = 2,$ $F^{P}_{P_1P_0} = {}^{0}a_1(q) - 0 = q - 2 = f_1(q).$

$2\delta < \underline{\dim} P < 4\delta$

$\underline{\dim}P(6,2) = 2\delta + \underline{\dim}P(0,2),$ $\underline{\dim}P'(6,2) = \delta + \underline{\dim}P'(0,2),$ $\underline{\dim}P''(6,2) = \delta + \underline{\dim}P''(0,2),$
$n = 3,$ $F^{P}_{P_1P_0} = {}^{3}a_0(q) - {}^{0}a_1(q) = q^2 - 3q + 3 = f_2(q).$

$\underline{\dim}P(6,4) = 2\delta + \underline{\dim}P(0,4),$ $\underline{\dim}P'(6,4) = \delta + \underline{\dim}P'(0,4),$ $\underline{\dim}P''(6,4) = \delta + \underline{\dim}P''(0,4),$
$n = 3,$ $F^{P}_{P_1P_0} = {}^{3}a_0(q) - {}^{0}a_1(q) = q^2 - 3q + 3 = f_2(q).$

$\underline{\dim}P(6,6) = 2\delta + \underline{\dim}P(0,6),$ $\underline{\dim}P'(6,3) = \delta + \underline{\dim}P'(0,6),$ $\underline{\dim}P''(6,6) = \delta + \underline{\dim}P''(0,6),$
$n = 3,$ $F^{P}_{P_1P_0} = {}^{3}a_0(q) - {}^{0}a_1(q) = q^2 - 3q + 3 = f_2(q).$

$\underline{\dim}P(8,2) = 2\delta + \underline{\dim}P(2,2),$ $\underline{\dim}P'(8,2) = \delta + \underline{\dim}P'(2,2),$ $\underline{\dim}P''(8,2) = \delta + \underline{\dim}P''(2,2),$
$n = 3,$ $F^{P}_{P_1P_0} = {}^{1}a_1(q) - {}^{2}a_0(q) = q^2 - 3q + 3 = f_2(q).$

$\underline{\dim}P(9,2) = 2\delta + \underline{\dim}P(3,2),$ $\underline{\dim}P'(9,2) = \delta + \underline{\dim}P'(3,2),$ $\underline{\dim}P''(9,2) = \delta + \underline{\dim}P''(3,2),$
$n = 4,$ $F^{P}_{P_1P_0} = {}^{2}a_1(q) - {}^{1}a_1(q) = f_3(q).$

$\underline{\dim}P(9,4) = 2\delta + \underline{\dim}P(3,4),$ $\underline{\dim}P'(9,4) = \delta + \underline{\dim}P'(3,4),$ $\underline{\dim}P''(9,4) = \delta + \underline{\dim}P''(3,4),$
$n = 4,$ $F^{P}_{P_1P_0} = {}^{2}a_1(q) - {}^{1}a_1(q) = f_3(q).$

$\underline{\dim}P(9,6) = 2\delta + \underline{\dim}P(3,6),$ $\underline{\dim}P'(9,6) = \delta + \underline{\dim}P'(3,6),$ $\underline{\dim}P''(9,6) = \delta + \underline{\dim}P''(3,6),$
$n = 4,$ $F^{P}_{P_1P_0} = {}^{2}a_1(q) - {}^{1}a_1(q) = f_3(q).$

$\underline{\dim}P(11,2) = 2\delta + \underline{\dim}P(5,2),$ $\underline{\dim}P'(11,2) = \delta + \underline{\dim}P'(5,2),$ $\underline{\dim}P''(11,2) = \delta + \underline{\dim}P''(5,2),$
$n = 4,$ $F^{P}_{P_1P_0} = {}^{0}a_2(q) - {}^{3}a_0(q) = f_3(q).$

B.3 Type $\widetilde{\mathbb{E}}_7$

$\underline{\dim} P < 2\delta$

$\underline{\dim}P(0,2) = \left(\begin{smallmatrix} &&&0&&& \\ 1&1&0&0&0&0&0 \end{smallmatrix}\right),$ $\underline{\dim}P'(0,2) = \left(\begin{smallmatrix} &&&0&&& \\ 1&0&0&0&0&0&0 \end{smallmatrix}\right),$ $\underline{\dim}P''(0,2) = \left(\begin{smallmatrix} &&&0&&& \\ 0&1&0&0&0&0&0 \end{smallmatrix}\right),$
$n = 1,$ $F^{P}_{P_1P_0} = 1 = f_0(q).$

$\underline{\dim}P(0,8) = \left(\begin{smallmatrix} &&&1&&& \\ 1&1&1&1&0&0&0 \end{smallmatrix}\right),$ $\underline{\dim}P'(0,8) = \left(\begin{smallmatrix} &&&0&&& \\ 1&0&0&0&0&0&0 \end{smallmatrix}\right),$ $\underline{\dim}P''(0,8) = \left(\begin{smallmatrix} &&&1&&& \\ 0&1&1&1&0&0&0 \end{smallmatrix}\right),$
$n = 1,$ $F^{P}_{P_1P_0} = 1 = f_0(q).$

$\underline{\dim}P(0,6) = \left(\begin{smallmatrix} &&&0&&& \\ 1&1&1&1&1&1&0 \end{smallmatrix}\right),$ $\underline{\dim}P'(0,6) = \left(\begin{smallmatrix} &&&0&&& \\ 1&0&0&0&0&0&0 \end{smallmatrix}\right),$ $\underline{\dim}P''(0,6) = \left(\begin{smallmatrix} &&&0&&& \\ 0&1&1&1&1&1&0 \end{smallmatrix}\right),$
$n = 1,$ $F^{P}_{P_1P_0} = 1 = f_0(q).$

$\underline{\dim}P(2,8) = \left(\begin{smallmatrix} &&&1&&& \\ 1&1&2&2&1&1&0 \end{smallmatrix}\right),$ $\underline{\dim}P'(2,8) = \left(\begin{smallmatrix} &&&0&&& \\ 0&0&1&0&0&0&0 \end{smallmatrix}\right),$ $\underline{\dim}P''(2,8) = \left(\begin{smallmatrix} &&&1&&& \\ 1&1&1&2&1&1&0 \end{smallmatrix}\right),$
$n = 1,$ $F^{P}_{P_1P_0} = {}^{1}a_0(q) - 0 = 1 = f_0(q).$

$\underline{\dim}P(3,2) = \left(\begin{smallmatrix}&&&1&&&\\1&1&1&2&1&0&0\end{smallmatrix}\right)$, $n = 1$,
$\underline{\dim}P'(3,2) = \left(\begin{smallmatrix}&&&0&&&\\0&0&0&1&0&0&0\end{smallmatrix}\right)$, $F^{P}_{P_1P_0} = {}^{1}a_0(q) - 0 = 1 = f_0(q)$.
$\underline{\dim}P''(3,2) = \left(\begin{smallmatrix}&&&1&&&\\1&1&1&1&1&0&0\end{smallmatrix}\right)$,

$\underline{\dim}P(3,6) = \left(\begin{smallmatrix}&&&1&&&\\1&1&2&3&2&1&0\end{smallmatrix}\right)$, $n = 1$,
$\underline{\dim}P'(3,6) = \left(\begin{smallmatrix}&&&0&&&\\0&0&0&1&0&0&0\end{smallmatrix}\right)$, $F^{P}_{P_1P_0} = {}^{1}a_0(q) - 0 = 1 = f_0(q)$.
$\underline{\dim}P''(3,6) = \left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&2&1&0\end{smallmatrix}\right)$,

$\underline{\dim}P(4,6) = \left(\begin{smallmatrix}&&&2&&&\\2&3&3&4&3&2&1\end{smallmatrix}\right)$, $n = 2$,
$\underline{\dim}P'(4,6) = \left(\begin{smallmatrix}&&&1&&&\\1&1&1&1&1&0&0\end{smallmatrix}\right)$, $F^{P}_{P_1P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q - 2 = f_1(q)$.
$\underline{\dim}P''(4,6) = \left(\begin{smallmatrix}&&&1&&&\\1&2&2&3&2&2&1\end{smallmatrix}\right)$,

$\underline{\dim}P(6,8) = \left(\begin{smallmatrix}&&&3&&&\\2&3&4&5&3&2&1\end{smallmatrix}\right)$, $n = 2$,
$\underline{\dim}P'(6,8) = \left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&1&1&1\end{smallmatrix}\right)$, $F^{P}_{P_1P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q - 2 = f_1(q)$.
$\underline{\dim}P''(6,8) = \left(\begin{smallmatrix}&&&2&&&\\1&2&2&3&2&1&0\end{smallmatrix}\right)$,

$\underline{\dim}P(7,6) = \left(\begin{smallmatrix}&&&3&&&\\2&3&4&6&4&2&1\end{smallmatrix}\right)$, $n = 2$,
$\underline{\dim}P'(7,6) = \left(\begin{smallmatrix}&&&1&&&\\0&1&1&2&1&0&0\end{smallmatrix}\right)$, $F^{P}_{P_1P_0} = {}^{0}a_1(q) - 0 = q - 2 = f_1(q)$.
$\underline{\dim}P''(7,6) = \left(\begin{smallmatrix}&&&2&&&\\2&2&3&4&3&2&1\end{smallmatrix}\right)$,

$\underline{\dim}P(8,2) = \left(\begin{smallmatrix}&&&2&&&\\2&3&4&5&4&3&1\end{smallmatrix}\right)$, $n = 2$,
$\underline{\dim}P'(8,2) = \left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&2&1&0\end{smallmatrix}\right)$, $F^{P}_{P_1P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q - 2 = f_1(q)$.
$\underline{\dim}P''(8,2) = \left(\begin{smallmatrix}&&&1&&&\\1&2&2&3&2&2&1\end{smallmatrix}\right)$,

$\underline{\dim}P(8,8) = \left(\begin{smallmatrix}&&&3&&&\\2&3&5&6&4&3&1\end{smallmatrix}\right)$, $n = 2$,
$\underline{\dim}P'(8,8) = \left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&2&1&0\end{smallmatrix}\right)$, $F^{P}_{P_1P_0} = {}^{2}a_0(q) - {}^{1}a_0(q) = q - 2 = f_1(q)$.
$\underline{\dim}P''(8,8) = \left(\begin{smallmatrix}&&&2&&&\\1&2&3&4&2&2&1\end{smallmatrix}\right)$,

$\underline{\dim}P(11,2) = \left(\begin{smallmatrix}&&&3&&&\\2&3&5&7&5&3&1\end{smallmatrix}\right)$, $n = 2$,
$\underline{\dim}P'(11,2) = \left(\begin{smallmatrix}&&&1&&&\\0&1&2&3&2&1&0\end{smallmatrix}\right)$, $F^{P}_{P_1P_0} = {}^{0}a_1(q) - 0 = q - 2 = f_1(q)$.
$\underline{\dim}P''(11,2) = \left(\begin{smallmatrix}&&&2&&&\\2&2&3&4&3&2&1\end{smallmatrix}\right)$,

$2\delta < \underline{\dim}P < 4\delta$

$\underline{\dim}P(12,2) = 2\delta + \underline{\dim}P(0,2)$, $n = 3$,
$\underline{\dim}P'(12,2) = \delta + \underline{\dim}P'(0,2)$, $F^{P}_{P_1P_0} = {}^{3}a_0(q) - {}^{0}a_1(q) = f_2(q)$.
$\underline{\dim}P''(12,2) = \delta + \underline{\dim}P''(0,2)$,

$\underline{\dim}P(12,8) = 2\delta + \underline{\dim}P(0,8)$, $n = 3$,
$\underline{\dim}P'(12,8) = \delta + \underline{\dim}P'(0,8)$, $F^{P}_{P_1P_0} = {}^{3}a_0(q) - {}^{0}a_1(q) = f_2(q)$.
$\underline{\dim}P''(12,8) = \delta + \underline{\dim}P''(0,8)$,

$\underline{\dim}P(12,6) = 2\delta + \underline{\dim}P(0,6)$, $n = 3$,
$\underline{\dim}P'(12,6) = \delta + \underline{\dim}P'(0,6)$, $F^{P}_{P_1P_0} = {}^{3}a_0(q) - {}^{0}a_1(q) = f_2(q)$.
$\underline{\dim}P''(12,6) = \delta + \underline{\dim}P''(0,6)$,

$\underline{\dim}P(14,8) = 2\delta + \underline{\dim}P(2,8)$, $n = 3$,
$\underline{\dim}P'(14,8) = \delta + \underline{\dim}P'(2,8)$, $F^{P}_{P_1P_0} = {}^{1}a_1(q) - {}^{2}a_0(q) = f_2(q)$.
$\underline{\dim}P''(14,8) = \delta + \underline{\dim}P''(2,8)$,

$\underline{\dim}P(15,2) = 2\delta + \underline{\dim}P(3,2)$, $n = 3$,
$\underline{\dim}P'(15,2) = \delta + \underline{\dim}P'(3,2)$, $F^{P}_{P_1P_0} = {}^{1}a_1(q) - {}^{2}a_0(q) = f_2(q)$.
$\underline{\dim}P''(15,2) = \delta + \underline{\dim}P''(3,2)$,

$\underline{\dim}P(15,6) = 2\delta + \underline{\dim}P(3,6)$, $n = 3$,
$\underline{\dim}P'(15,6) = \delta + \underline{\dim}P'(3,6)$, $F^P_{P_1P_0} = {}^1a_1(q) - {}^2a_0(q) = f_2(q)$.
$\underline{\dim}P''(15,6) = \delta + \underline{\dim}P''(3,6)$,

$\underline{\dim}P(16,6) = 2\delta + \underline{\dim}P(4,6)$, $n = 4$,
$\underline{\dim}P'(16,6) = \delta + \underline{\dim}P'(4,6)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$.
$\underline{\dim}P''(16,6) = \delta + \underline{\dim}P''(4,6)$,

$\underline{\dim}P(18,8) = 2\delta + \underline{\dim}P(6,8)$, $n = 4$,
$\underline{\dim}P'(18,8) = \delta + \underline{\dim}P'(6,8)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$.
$\underline{\dim}P''(18,8) = \delta + \underline{\dim}P''(6,8)$,

$\underline{\dim}P(19,6) = 2\delta + \underline{\dim}P(7,6)$, $n = 4$,
$\underline{\dim}P'(19,6) = \delta + \underline{\dim}P'(7,6)$, $F^P_{P_1P_0} = {}^0a_2(q) - {}^3a_0(q) = f_3(q)$.
$\underline{\dim}P''(19,6) = \delta + \underline{\dim}P''(7,6)$,

$\underline{\dim}P(20,2) = 2\delta + \underline{\dim}P(8,2)$, $n = 4$,
$\underline{\dim}P'(20,2) = \delta + \underline{\dim}P'(8,2)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$.
$\underline{\dim}P''(20,2) = \delta + \underline{\dim}P''(8,2)$,

$\underline{\dim}P(20,8) = 2\delta + \underline{\dim}P(8,8)$, $n = 4$,
$\underline{\dim}P'(20,8) = \delta + \underline{\dim}P'(8,8)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$.
$\underline{\dim}P''(20,8) = \delta + \underline{\dim}P''(8,8)$,

$\underline{\dim}P(23,2) = 2\delta + \underline{\dim}P(11,2)$, $n = 4$,
$\underline{\dim}P'(23,2) = \delta + \underline{\dim}P'(11,2)$, $F^P_{P_1P_0} = {}^0a_2(q) - {}^3a_0(q) = f_3(q)$.
$\underline{\dim}P''(23,2) = \delta + \underline{\dim}P''(11,2)$,

B.4 Type $\widetilde{\mathbb{E}}_8$

$\underline{\dim}P < 2\delta$

$\underline{\dim}P(0,2) = (1\;1\;0\;0\;0\;\overset{0}{0}\;0\;0)$, $n = 1$
$\underline{\dim}P'(0,2) = (1\;0\;0\;0\;0\;\overset{0}{0}\;0\;0)$, $F^P_{P_1P_0} = 1 = f_0(q)$.
$\underline{\dim}P''(0,2) = (0\;1\;0\;0\;0\;\overset{0}{0}\;0\;0)$,

$\underline{\dim}P(0,8) = (1\;1\;1\;1\;1\;\overset{0}{1}\;1\;1)$, $n = 1$
$\underline{\dim}P'(0,8) = (1\;0\;0\;0\;0\;\overset{0}{0}\;0\;0)$, $F^P_{P_1P_0} = 1 = f_0(q)$.
$\underline{\dim}P''(0,8) = (0\;1\;1\;1\;1\;\overset{0}{1}\;1\;1)$,

$\underline{\dim}P(3,8) = (1\;1\;1\;2\;2\;\overset{1}{2}\;1\;1)$, $n = 1$
$\underline{\dim}P'(3,8) = (0\;0\;1\;0\;0\;\overset{0}{0}\;0\;0)$, $F^P_{P_1P_0} = {}^1a_0(q) - 0 = 1 = f_0(q)$.
$\underline{\dim}P''(3,8) = (1\;1\;0\;2\;2\;\overset{1}{2}\;1\;1)$,

$\underline{\dim}P(5,2) = (1\;1\;1\;1\;1\;\overset{1}{2}\;1\;0)$, $n = 1$
$\underline{\dim}P'(5,2) = (0\;0\;0\;0\;0\;\overset{0}{1}\;0\;0)$, $F^P_{P_1P_0} = {}^1a_0(q) - 0 = 1 = f_0(q)$.
$\underline{\dim}P''(5,2) = (1\;1\;1\;1\;1\;\overset{1}{1}\;1\;0)$,

$\underline{\dim}P(5,8) = (1\;1\;2\;2\;2\;\overset{1}{3}\;2\;1)$, $n = 1$
$\underline{\dim}P'(5,8) = (0\;0\;0\;0\;0\;\overset{0}{1}\;0\;0)$, $F^P_{P_1P_0} = {}^1a_0(q) - 0 = 1 = f_0(q)$.
$\underline{\dim}P''(5,8) = (1\;1\;2\;2\;2\;\overset{1}{2}\;2\;1)$,

$\underline{\dim} P(8,8) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&1&2&3&3&4&2&1 \end{smallmatrix}\right)$, $n = 1$

$\underline{\dim} P'(8,8) = \left(\begin{smallmatrix} &&&&&1&& \\ 0&0&1&1&1&1&0&0 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^1a_0(q) - 0 = 1 = f_0(q)$.

$\underline{\dim} P''(8,8) = \left(\begin{smallmatrix} &&&&&1&& \\ 1&1&1&2&2&3&2&1 \end{smallmatrix}\right)$,

$\underline{\dim} P(9,2) = \left(\begin{smallmatrix} &&&&&1&& \\ 1&1&1&2&3&3&2&1 \end{smallmatrix}\right)$, $n = 1$

$\underline{\dim} P'(9,2) = \left(\begin{smallmatrix} &&&&&0&& \\ 0&0&0&1&1&1&1&0 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^1a_0(q) - 0 = 1 = f_0(q)$.

$\underline{\dim} P''(9,2) = \left(\begin{smallmatrix} &&&&&1&& \\ 1&1&1&1&2&2&1&1 \end{smallmatrix}\right)$,

$\underline{\dim} P(14,2) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&1&2&3&4&5&3&1 \end{smallmatrix}\right)$, $n = 1$

$\underline{\dim} P'(14,2) = \left(\begin{smallmatrix} &&&&&1&& \\ 0&0&1&1&2&2&1&0 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^1a_0(q) - 0 = 1 = f_0(q)$.

$\underline{\dim} P''(14,2) = \left(\begin{smallmatrix} &&&&&1&& \\ 1&1&1&2&2&3&2&1 \end{smallmatrix}\right)$,

$\underline{\dim} P(15,2) = \left(\begin{smallmatrix} &&&&&3&& \\ 2&3&3&4&5&6&4&2 \end{smallmatrix}\right)$, $n = 2$

$\underline{\dim} P'(15,2) = \left(\begin{smallmatrix} &&&&&1&& \\ 1&1&1&2&2&3&2&1 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^2a_0(q) - {}^1a_0(q) = q - 2 = f_1(q)$.

$\underline{\dim} P''(15,2) = \left(\begin{smallmatrix} &&&&&3&& \\ 1&2&2&2&3&2&1&2 \end{smallmatrix}\right)$,

$\underline{\dim} P(15,8) = \left(\begin{smallmatrix} &&&&&3&& \\ 2&3&4&5&6&7&5&3 \end{smallmatrix}\right)$, $n = 2$

$\underline{\dim} P'(15,8) = \left(\begin{smallmatrix} &&&&&1&& \\ 1&1&1&2&2&3&2&1 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^2a_0(q) - {}^1a_0(q) = q - 2 = f_1(q)$.

$\underline{\dim} P''(15,8) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&2&3&3&4&4&3&2 \end{smallmatrix}\right)$,

$\underline{\dim} P(18,8) = \left(\begin{smallmatrix} &&&&&4&& \\ 2&3&4&6&7&8&5&3 \end{smallmatrix}\right)$, $n = 2$

$\underline{\dim} P'(18,8) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&1&2&3&3&3&2&1 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^2a_0(q) - {}^1a_0(q) = q - 2 = f_1(q)$.

$\underline{\dim} P''(18,8) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&2&2&3&4&5&3&2 \end{smallmatrix}\right)$,

$\underline{\dim} P(20,2) = \left(\begin{smallmatrix} &&&&&4&& \\ 2&3&4&5&6&8&5&2 \end{smallmatrix}\right)$, $n = 2$

$\underline{\dim} P'(20,2) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&1&2&2&3&4&2&1 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^2a_0(q) - {}^1a_0(q) = q - 2 = f_1(q)$.

$\underline{\dim} P''(20,2) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&2&2&3&3&4&3&1 \end{smallmatrix}\right)$,

$\underline{\dim} P(20,8) = \left(\begin{smallmatrix} &&&&&4&& \\ 2&3&5&6&7&9&6&3 \end{smallmatrix}\right)$, $n = 2$

$\underline{\dim} P'(20,8) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&1&2&2&3&4&2&1 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^2a_0(q) - {}^1a_0(q) = q - 2 = f_1(q)$.

$\underline{\dim} P''(20,8) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&2&3&4&4&5&4&2 \end{smallmatrix}\right)$,

$\underline{\dim} P(23,8) = \left(\begin{smallmatrix} &&&&&5&& \\ 2&3&5&7&8&10&6&3 \end{smallmatrix}\right)$, $n = 2$

$\underline{\dim} P'(23,8) = \left(\begin{smallmatrix} &&&&&2&& \\ 0&1&2&3&3&4&2&1 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^0a_1(q) - 0 = q - 2 = f_1(q)$.

$\underline{\dim} P''(23,8) = \left(\begin{smallmatrix} &&&&&3&& \\ 2&2&3&4&5&6&4&2 \end{smallmatrix}\right)$,

$\underline{\dim} P(24,8) = \left(\begin{smallmatrix} &&&&&5&& \\ 2&4&5&7&9&10&7&3 \end{smallmatrix}\right)$, $n = 2$

$\underline{\dim} P'(24,8) = \left(\begin{smallmatrix} &&&&&2&& \\ 1&1&2&3&4&4&3&1 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^2a_0(q) - {}^1a_0(q) = q - 2 = f_1(q)$.

$\underline{\dim} P''(24,8) = \left(\begin{smallmatrix} &&&&&3&& \\ 1&3&3&4&5&6&4&2 \end{smallmatrix}\right)$,

$\underline{\dim} P(29,2) = \left(\begin{smallmatrix} &&&&&5&& \\ 2&3&5&7&9&11&7&3 \end{smallmatrix}\right)$, $n = 2$

$\underline{\dim} P'(29,2) = \left(\begin{smallmatrix} &&&&&2&& \\ 0&1&2&3&4&5&3&1 \end{smallmatrix}\right)$, $F^P_{P_1P_0} = {}^0a_1(q) - 0 = q - 2 = f_1(q)$.

$\underline{\dim} P''(29,2) = \left(\begin{smallmatrix} &&&&&3&& \\ 2&2&3&4&5&6&4&2 \end{smallmatrix}\right)$,

$2\delta < \underline{\dim} P < 4\delta$

$\underline{\dim} P(30,2) = 2\delta + \underline{\dim} P(0,2)$, $n = 3$,

$\underline{\dim} P'(30,2) = \delta + \underline{\dim} P'(0,2)$, $F^P_{P_1P_0} = {}^3a_0(q) - {}^0a_1(q) = f_2(q)$.

$\underline{\dim} P''(30,2) = \delta + \underline{\dim} P''(0,2)$,

$\underline{\dim} P(30,8) = 2\delta + \underline{\dim} P(0,8)$, $n = 3$,

$\underline{\dim} P'(30,8) = \delta + \underline{\dim} P'(0,8)$, $F^P_{P_1P_0} = {}^3a_0(q) - {}^0a_1(q) = f_2(q)$.

$\underline{\dim} P''(30,8) = \delta + \underline{\dim} P''(0,8)$,

$\underline{\dim} P(33,8) = 2\delta + \underline{\dim} P(3,8)$, $n = 3$, $\underline{\dim} P'(33,8) = \delta + \underline{\dim} P'(3,8)$, $F^P_{P_1P_0} = {}^1a_1(q) - {}^2a_0(q) = f_2(q)$. $\underline{\dim} P''(33,8) = \delta + \underline{\dim} P''(3,8)$,

$\underline{\dim} P(35,2) = 2\delta + \underline{\dim} P(5,2)$, $n = 3$, $\underline{\dim} P'(35,2) = \delta + \underline{\dim} P'(5,2)$, $F^P_{P_1P_0} = {}^1a_1(q) - {}^2a_0(q) = f_2(q)$. $\underline{\dim} P''(35,2) = \delta + \underline{\dim} P''(5,2)$,

$\underline{\dim} P(35,8) = 2\delta + \underline{\dim} P(5,8)$, $n = 3$, $\underline{\dim} P'(35,8) = \delta + \underline{\dim} P'(5,8)$, $F^P_{P_1P_0} = {}^1a_1(q) - {}^2a_0(q) = f_2(q)$. $\underline{\dim} P''(35,8) = \delta + \underline{\dim} P''(5,8)$,

$\underline{\dim} P(38,8) = 2\delta + \underline{\dim} P(8,8)$, $n = 3$, $\underline{\dim} P'(38,8) = \delta + \underline{\dim} P'(8,8)$, $F^P_{P_1P_0} = {}^1a_1(q) - {}^2a_0(q) = f_2(q)$. $\underline{\dim} P''(38,8) = \delta + \underline{\dim} P''(8,8)$,

$\underline{\dim} P(39,2) = 2\delta + \underline{\dim} P(9,2)$, $n = 3$, $\underline{\dim} P'(39,2) = \delta + \underline{\dim} P'(9,2)$, $F^P_{P_1P_0} = {}^1a_1(q) - {}^2a_0(q) = f_2(q)$. $\underline{\dim} P''(39,2) = \delta + \underline{\dim} P''(9,2)$,

$\underline{\dim} P(44,2) = 2\delta + \underline{\dim} P(14,2)$, $n = 3$, $\underline{\dim} P'(44,2) = \delta + \underline{\dim} P'(14,2)$, $F^P_{P_1P_0} = {}^1a_1(q) - {}^2a_0(q) = f_2(q)$. $\underline{\dim} P''(44,2) = \delta + \underline{\dim} P''(14,2)$,

$\underline{\dim} P(45,2) = 2\delta + \underline{\dim} P(15,2)$, $n = 4$, $\underline{\dim} P'(45,2) = \delta + \underline{\dim} P'(15,2)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$. $\underline{\dim} P''(45,2) = \delta + \underline{\dim} P''(15,2)$,

$\underline{\dim} P(45,8) = 2\delta + \underline{\dim} P(15,8)$, $n = 4$, $\underline{\dim} P'(45,8) = \delta + \underline{\dim} P'(15,8)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$. $\underline{\dim} P''(45,8) = \delta + \underline{\dim} P''(15,8)$,

$\underline{\dim} P(48,8) = 2\delta + \underline{\dim} P(18,8)$, $n = 4$, $\underline{\dim} P'(48,8) = \delta + \underline{\dim} P'(18,8)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$. $\underline{\dim} P''(48,8) = \delta + \underline{\dim} P''(18,8)$,

$\underline{\dim} P(50,2) = 2\delta + \underline{\dim} P(20,2)$, $n = 4$, $\underline{\dim} P'(50,2) = \delta + \underline{\dim} P'(20,2)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$. $\underline{\dim} P''(50,2) = \delta + \underline{\dim} P''(20,2)$,

$\underline{\dim} P(50,8) = 2\delta + \underline{\dim} P(20,8)$, $n = 4$, $\underline{\dim} P'(50,8) = \delta + \underline{\dim} P'(20,8)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$. $\underline{\dim} P''(50,8) = \delta + \underline{\dim} P''(20,8)$,

$\underline{\dim} P(53,8) = 2\delta + \underline{\dim} P(23,8)$, $n = 4$, $\underline{\dim} P'(53,8) = \delta + \underline{\dim} P'(23,8)$, $F^P_{P_1P_0} = {}^0a_2(q) - {}^3a_0(q) = f_3(q)$. $\underline{\dim} P''(53,8) = \delta + \underline{\dim} P''(23,8)$,

$\underline{\dim} P(54,8) = 2\delta + \underline{\dim} P(24,8)$, $n = 4$, $\underline{\dim} P'(54,8) = \delta + \underline{\dim} P'(24,8)$, $F^P_{P_1P_0} = {}^2a_1(q) - {}^1a_1(q) = f_3(q)$. $\underline{\dim} P''(54,8) = \delta + \underline{\dim} P''(24,8)$,

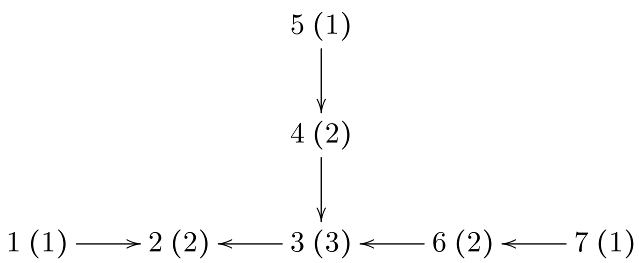
$\underline{\dim} P(59,2) = 2\delta + \underline{\dim} P(29,2)$, $n = 4$, $\underline{\dim} P'(59,2) = \delta + \underline{\dim} P'(29,2)$, $F^P_{P_1P_0} = {}^0a_2(q) - {}^3a_0(q) = f_3(q)$. $\underline{\dim} P''(59,2) = \delta + \underline{\dim} P''(29,2)$,

Appendix C

Proof of Theorem 2.10

C.1 The $\widetilde{\mathbb{E}}_6$ case

Due to symmetry we can suppose that the quiver of type $\widetilde{\mathbb{E}}_6$ has the following orientation) the vertex 2 being its unique sink, thus $P_0 = S_2$):



We have 3 non-homogeneous tubes: $\mathcal{T}_\infty$ of rank 2 and $\mathcal{T}_0, \mathcal{T}_1$ of rank 3. The regular-simples of these tubes are:

- ${}^1R(1,\infty)$ of dimension $\left(\begin{smallmatrix} &&0&&\\ &&1&&\\ 1&1&1&1&0\end{smallmatrix}\right)$, ${}^2R(1,\infty)$ of dimension $\left(\begin{smallmatrix} &&1&&\\ &&1&&\\ 0&1&2&1&1\end{smallmatrix}\right)$;
- ${}^1R(1,0)$ of dimension $\left(\begin{smallmatrix} &&0&&\\ &&0&&\\ 0&0&1&1&0\end{smallmatrix}\right)$, ${}^2R(1,0)$ of dimension $\left(\begin{smallmatrix} &&0&&\\ &&1&&\\ 0&1&1&1&1\end{smallmatrix}\right)$ and ${}^3R(1,0)$ of dimension $\left(\begin{smallmatrix} &&1&&\\ &&1&&\\ 1&1&1&0&0\end{smallmatrix}\right)$;
- ${}^1R(1,1)$ of dimension $\left(\begin{smallmatrix} &&0&&\\ &&1&&\\ 0&0&1&0&0\end{smallmatrix}\right)$, ${}^2R(1,1)$ of dimension $\left(\begin{smallmatrix} &&1&&\\ &&1&&\\ 0&1&1&1&0\end{smallmatrix}\right)$ and ${}^3R(1,1)$ of dimension $\left(\begin{smallmatrix} &&0&&\\ &&0&&\\ 1&1&1&1&1\end{smallmatrix}\right)$.

As in Appendix A, we will use the notation $P = P(t,i)$, meaning that we consider the τ^{-t} shift of the projective corresponding to vertex i.

Steps 1 and 2

We list the dimensions $\sigma \in \Sigma_Q$, P_σ, the generic form of the elements in $\mathcal{P}_{\sigma+2\delta}$ or $\mathcal{P}_\sigma$, and the considered corresponding generic orthogonal exceptional pair:

(1) $\sigma_1 = \left(\begin{smallmatrix} &&0&&\\ &&0&&\\ 0&0&0&0&0\end{smallmatrix}\right) \in \Sigma''_Q$,
$P_{\sigma_1} = 0$ does not possess an orthogonal exceptional pair,
$P_{\sigma_1+2\delta} = P(6,2)$, $\underline{\dim} P(6,2) = \left(\begin{smallmatrix} &&2&&\\ &&4&&\\ 2&5&6&4&2\end{smallmatrix}\right)$, $\underline{\dim} P'(6,2) = \left(\begin{smallmatrix} &&1&&\\ &&2&&\\ 0&2&3&2&1\end{smallmatrix}\right)$, $\underline{\dim} P''(6,2) = \left(\begin{smallmatrix} &&1&&\\ &&2&&\\ 2&3&3&2&1\end{smallmatrix}\right)$,
$\mathcal{P}_{\sigma_1+2\delta} = \{[P(6n+6,2)] | n \in \mathbb{N}\}$, $P'(6n+6,2) = P'(6,2)(+\delta)$, $P''(6n+6,2) = P''(6,2)(+\delta)$;

(2) $\sigma_2 = \left(\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}\right) \in \Sigma''_Q$,
$P_{\sigma_2} = P(0,4)$ does not possess an orthogonal exceptional pair,
$P_{\sigma_2+2\delta} = P(6,4)$, $\underline{\dim}P(6,4) = \left(\begin{smallmatrix} & & 2 & & \\ & & 5 & & \\ 2 & 5 & 7 & 4 & 2 \end{smallmatrix}\right)$, $\underline{\dim}P'(6,4) = \left(\begin{smallmatrix} & & 0 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}\right)$, $\underline{\dim}P''(6,4) = \left(\begin{smallmatrix} & & 2 & & \\ & & 3 & & \\ 1 & 3 & 4 & 2 & 1 \end{smallmatrix}\right)$,
$\mathcal{P}_{\sigma_2+2\delta} = \{[P(6n+6,4)] | n \in \mathbb{N}\}$, $P'(6n+6,4) = P'(6,4)(+\delta)$, $P''(6n+6,4) = P''(6,4)(+\delta)$;

(3) $\sigma_3 = \left(\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 1 & 1 & 2 & 1 & 0 \end{smallmatrix}\right) \in \Sigma'_Q$,
$P_{\sigma_3} = P(1,4)$, $\underline{\dim}P(1,4) = \left(\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 1 & 2 & 2 & 1 & 0 \end{smallmatrix}\right)$, $\underline{\dim}P'(1,4) = \left(\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 0 & 1 & 1 & 0 & 0 \end{smallmatrix}\right)$, $\underline{\dim}P''(1,4) = \left(\begin{smallmatrix} & & 0 & & \\ & & 0 & & \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}\right)$,
$\mathcal{P}_{\sigma_3} = \{[P(6n+1,4)] | n \in \mathbb{N}\}$, $P'(6n+1,4) = P'(1,4)(+\delta)$, $P''(6n+1,4) = P''(1,4)(+\delta)$;

(4) $\sigma_4 = \left(\begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}\right) \in \Sigma'_Q$,
$P_{\sigma_4} = P(3,2)$, $\underline{\dim}P(3,2) = \left(\begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 3 & 3 & 2 & 1 \end{smallmatrix}\right)$, $\underline{\dim}P'(3,2) = \left(\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix}\right)$, $\underline{\dim}P''(3,2) = \left(\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 1 & 2 & 2 & 1 & 1 \end{smallmatrix}\right)$,
$\mathcal{P}_{\sigma_4} = \{[P(6n+3,2)] | n \in \mathbb{N}\}$, $P'(6n+3,2) = P'(3,2)(+\delta)$, $P''(6n+3,2) = P''(3,2)(+\delta)$;

(5) $\sigma_5 = \left(\begin{smallmatrix} & & 1 & & \\ & & 3 & & \\ 1 & 2 & 4 & 2 & 1 \end{smallmatrix}\right) \in \Sigma'_Q$,
$P_{\sigma_5} = P(3,4)$, $\underline{\dim}P(3,4) = \left(\begin{smallmatrix} & & 1 & & \\ & & 3 & & \\ 1 & 3 & 4 & 2 & 1 \end{smallmatrix}\right)$, $\underline{\dim}P'(3,4) = \left(\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 1 & 2 & 1 & 1 \end{smallmatrix}\right)$, $\underline{\dim}P''(3,4) = \left(\begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 2 & 1 & 0 \end{smallmatrix}\right)$,
$\mathcal{P}_{\sigma_5} = \{[P(6n+3,4)] | n \in \mathbb{N}\}$, $P'(6n+3,4) = P'(3,4)(+\delta)$, $P''(6n+3,4) = P''(3,4)(+\delta)$;

(6) $\sigma_6 = \left(\begin{smallmatrix} & & 2 & & \\ & & 3 & & \\ 2 & 3 & 5 & 3 & 1 \end{smallmatrix}\right) \in \Sigma'_Q$,
$P_{\sigma_6} = P(4,4)$, $\underline{\dim}P(4,4) = \left(\begin{smallmatrix} & & 2 & & \\ & & 3 & & \\ 2 & 4 & 5 & 3 & 1 \end{smallmatrix}\right)$, $\underline{\dim}P'(4,4) = \left(\begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 2 & 1 & 0 \end{smallmatrix}\right)$, $\underline{\dim}P''(4,4) = \left(\begin{smallmatrix} & & 1 & & \\ & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}\right)$,
$\mathcal{P}_{\sigma_6} = \{[P(6n+4,4)] | n \in \mathbb{N}\}$, $P'(6n+4,4) = P'(4,4)(+\delta)$, $P''(6n+4,4) = P''(4,4)(+\delta)$.

Note that for σ_1, σ_4 the regular factor R has dimension $\underline{\dim}R = \underline{\dim}P - \underline{\dim}P_0$ a multiple of δ, thus it can be taken from any of the 3 non-homogeneous tubes. For the other σ-s the dimension of R determines uniquely its non-homogeneous tube.

Step 3

We follow now a case by case computation for each σ, computing $F^P_{RP_0} = S_g(R, P', P'', P_0) - S_g(R, P'', P', P_0) = S_1 - S_2$.

The case $\sigma_1 = (0,0,0,0,0,0,0)$

For $P_{\sigma_1} = P_0$ there is no orthogonal exceptional preprojective pair and trivially $F^{P_{\sigma_1}}_{0P_0} = 1$.

For $P = P(6n+6,2) = P_{\sigma_1}(+2(n+1)\delta)$ we have $\underline{\dim}R = (2n+2)\delta$. Thus R can be taken from any non-homogeneous tube.

• R is from the tube of rank 2

Without loss of generality we may suppose that $R = {}^1R(4n+4,\infty)$ meaning that its regular-socle is ${}^1R(1,\infty)$ and regular-top is ${}^2R(1,\infty)$. The other case, when $R = {}^2R(4n+4,\infty)$, will yield the same result. We have $\underline{\dim}P' = n\delta + (0,2,3,2,1,2,1)$, $\underline{\dim}P'' = n\delta + (2,3,3,2,1,2,1)$ and $R_{P'}(1,\infty) = {}^2R(1,\infty)$, $R_{P''}(1,\infty) = {}^1R(1,\infty)$, thus $\delta_{\mathrm{top}R, R_{P'}(1,\infty)} = 1$ and $\delta_{\mathrm{top}R, R_{P''}(1,\infty)} = 0$.

Since top$R$ = ${}^2R(1,\infty)$, there are no possible regular indecomposable pairs (R'',R') in $(**)$ such that $0 \to R'' \to R \to R' \to 0$ is exact and topR' = ${}^1R(1,\infty)$. Thus we have

$$S_1 = f_{2n+2}(q).$$

The possible regular indecomposable pairs (R'',R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim}R'' > \underline{\dim}P''$, top$R'' = R_{P''}(1,\infty) = {}^1R(1,\infty)$ and top$R' = R_{P'}(1,\infty) = {}^2R(1,\infty)$ are $({}^1R(2t+1,\infty), {}^2R(4n-2t+3,\infty))$, where $t \in \{n+1,\dots,2n+1\}$. Thus we have

$$\begin{aligned} S_2 &= \frac{1}{\alpha_{{}^1R(4n+4,\infty)}} \sum_{t=n+1}^{2n+1} \alpha_{{}^1R(2t+1,\infty)} f_{2(t-n-1)}(q) q^{2n+1-t} \alpha_{{}^2R(4n-2t+3,\infty)} \\ &= (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q). \end{aligned}$$

This means that the Ringel-Hall polynomial in this case is:

$$F^P_{RP_0} = S_1 - S_2 = f_{2n+2}(q) - (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q) = h_{2n+2}(q).$$

- **R is from a tube of rank 3**

(1) Suppose that $R = {}^1R(6n+6,0)$, meaning that its regular-socle is ${}^1R(1,0)$ and regular-top is ${}^3R(1,0)$. Also, $\underline{\dim}P' = n\delta + (0,2,3,2,1,2,1)$, $\underline{\dim}P'' = n\delta + (2,3,3,2,1,2,1)$ and $R_{P'}(1,0) = {}^2R(1,0)$, $R_{P''}(1,0) = {}^3R(1,0)$, thus $\delta_{\text{top}R, R_{P'}(1,0)} = 0$ and $\delta_{\text{top}R, R_{P''}(1,0)} = 1$.

The possible regular indecomposable pairs (R'',R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim}R'' > \underline{\dim}P'$, top$R'' = R_{P'}(1,0) = {}^2R(1,0)$ and top$R' = R_{P''}(1,0) = {}^3R(1,0)$ are $({}^1R(3t+2,0), {}^3R(6n-3t+4,0))$, where $t \in \{n+1,\dots,2n+1\}$. Thus we have

$$\begin{aligned} S_1 &= \frac{1}{\alpha_{{}^1R(6n+6,0)}} \sum_{t=n+1}^{2n+1} \alpha_{{}^1R(3t+2,0)} f_{2(t-n)-1}(q) q^{2n-t+1} \alpha_{{}^3R(6n-3t+4,0)} \\ &= (q-1)\sum_{s=0}^{n} q^s f_{2n-2s+1}(q). \end{aligned}$$

Since top$R$ = ${}^3R(1,0)$, there are no possible regular indecomposable pairs (R'',R') in $(**)$ such that $0 \to R'' \to R \to R' \to 0$ is exact and top$R' = R_{P'}(1,0) = {}^2R(1,0)$. Thus we have

$$S_2 = f_{2n+1}(q).$$

This means that the Ringel-Hall polynomial in this case is:

$$F^P_{RP_0} = S_1 - S_2 = (q-1)\sum_{s=0}^{n} q^s f_{2n-2s+1}(q) - f_{2n+1}(q) = h_{2n+2}(q).$$

(2) Suppose that $R = {}^2R(6n+6,0)$, meaning that its regular-socle is ${}^2R(1,0)$ and regular-top is ${}^1R(1,0)$.

In case $F^{P}_{RP_0} \neq 0$, the indecomposable P must project to the regular-top of R, thus $\langle \underline{\dim} P, \underline{\dim}\, {}^1R(1,0)\rangle = 0$ must be nonzero, a contradiction. So, the Ringel-Hall polynomial in this case is $F^{P}_{RP_0} = 0$.

(3) Suppose that $R = {}^3R(6n+6,0)$, meaning that its regular-socle is ${}^3R(1,0)$ and regular-top is ${}^2R(1,0)$. Also, $\delta_{\mathrm{top}R, R_{P'}(1,0)} = 1$ and $\delta_{\mathrm{top}R, R_{P''}(1,0)} = 0$.

Since $\mathrm{top}R = {}^2R(1,0)$, there are no possible regular indecomposable pairs (R'', R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact and $\mathrm{top}R' = R_{P'}(1,0) = {}^3R(1,0)$. Thus we have

$$S_1 = f_{2n+2}(q).$$

The possible regular indecomposable pairs (R'', R') in $(**)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim} R'' > \underline{\dim} P''$, $\mathrm{top}R'' = R_{P''}(1,0) = {}^3R(1,0)$ and $\mathrm{top}R' = R_{P'}(1,0) = {}^2R(1,0)$ are $({}^3R(3t+1,0), {}^1R(6n-3t+5,0))$, where $t \in \{n+1, \dots, 2n+1\}$. Thus we have

$$\begin{aligned} S_2 &= \frac{1}{\alpha_{{}^1R(6n+6,0)}} \sum_{t=n+1}^{2n+1} \alpha_{{}^3R(3t+1,0)} f_{2(t-n-1)}(q) q^{2n-t+1} \alpha_{{}^1R(6n-3t+5,0)} \\ &= (q-1) \sum_{s=0}^{n} q^s f_{2n-2s}(q). \end{aligned}$$

This means that the Ringel-Hall polynomial in this case is:

$$F^{P}_{RP_0} = S_1 - S_2 = f_{2n+2}(q) - (q-1) \sum_{s=0}^{n} q^s f_{2n-2s}(q) = h_{2n+2}(q).$$

(4) Suppose that $R = {}^1R(6n+6,1)$, meaning that its regular-socle is ${}^1R(1,1)$ and regular-top is ${}^3R(1,1)$. Also, $R_{P'}(1,1) = {}^2R(1,1)$, $R_{P''}(1,1) = {}^3R(1,1)$. By a similar argumentation as in case (1) we obtain that

$$F^{P}_{RP_0} = h_{2n+2}(q).$$

(5) Suppose that $R = {}^2R(6n+6,1)$, meaning that its regular-socle is ${}^2R(1,1)$ and regular-top is ${}^1R(1,1)$. By a similar argumentation as in case (2) we obtain that

$$F^{P}_{RP_0} = 0.$$

(6) Suppose that $R = {}^3R(6n+6,1)$, meaning that its regular-socle is ${}^3R(1,1)$ and regular-top is ${}^2R(1,1)$. By a similar argumentation as in case (3) we obtain that

$$F^{P}_{RP_0} = h_{2n+2}(q).$$

The case $\sigma_2 = (0,0,1,1,0,0,0)$

For $P_{\sigma_2} = P(0,4)$ there is no orthogonal exceptional preprojective pair, and trivially $F^{P_{\sigma_2}}_{{}^1R(1,1)P_0} = 1$.

For $P = P(6n+6,4) = P_{\sigma_2}(+2(n+1)\delta)$ we have $\underline{\dim}R = (2n+2)\delta + (0,0,1,1,0,0,0)$, thus $\underline{\dim}R = (2n+2)\delta + \underline{\dim}\,{}^1R(1,1)$, so $R = {}^1R(6n+7,1)$. Also, $\underline{\dim}P' = n\delta + (1,2,2,2,1,1,0)$, $\underline{\dim}P'' = n\delta + (1,2,3,1,1,2,1)$ and $R_{P'}(1,1) = {}^1R(1,1)$, $R_{P''}(1,1) = {}^2R(1,1)$, thus $\delta_{\mathrm{top}R,R_{P'}(1,1)} = 1$ and $\delta_{\mathrm{top}R,R_{P''}(1,1)} = 0$.

Since $\mathrm{top}R = {}^1R(1,1)$, there are no possible regular indecomposable pairs (R'',R') in $(**)$ such that $0 \to R'' \to R \to R' \to 0$ is exact and $\mathrm{top}R' = R_{P''}(1,1) = {}^2R(1,1)$. Thus we have

$$S_1 = f_{2n+2}(q).$$

The possible regular indecomposable pairs (R'',R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim}R'' > \underline{\dim}P'$, $\mathrm{top}R'' = R_{P''}(1,1) = {}^2R(1,1)$ and $\mathrm{top}R' = R_{P'}(1,1) = {}^1R(1,1)$ are $({}^1R(3t+2,1), {}^3R(6n-3t+5,1))$, where $t \in \{n+1,\dots,2n+1\}$. Thus we have

$$\begin{aligned} S_2 &= \frac{1}{\alpha_{1_{R(6n+7,1)}}} \sum_{t=n+1}^{2n+1} \alpha_{1_{R(3t+2,1)}} f_{2(t-n-1)}(q) q^{2n-t+2} \alpha_{3_{R(6n-3t+5,1)}} \\ &= (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q). \end{aligned}$$

This means that the Ringel-Hall polynomial in this case is:

$$F^P_{RP_0} = S_1 - S_2 = f_{2n+2}(q) - (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q) = h_{2n+2}(q).$$

The case $\sigma_3 = (1,1,2,1,1,1,0)$

In this case $P = P(6n+1,4) = P_{\sigma_3}(+2n\delta)$, $\underline{\dim}R = 2n\delta + (1,2,2,1,1,1,0)$, thus $\underline{\dim}R = 2n\delta + \underline{\dim}\,{}^3R(1,0) + \underline{\dim}\,{}^1R(1,0)$, so $R = {}^3R(6n+2,0)$. Also, $\underline{\dim}P' = n\delta + (0,1,1,1,1,0,0)$, $\underline{\dim}P'' = n\delta + (1,1,1,0,0,1,0)$ and $R_{P'}(1,0) = {}^3R(1,0)$, $R_{P''}(1,0) = {}^1R(1,0)$, thus $\delta_{\mathrm{top}R,R_{P'}(1,0)} = 0$ and $\delta_{\mathrm{top}R,R_{P''}(1,0)} = 1$.

The possible regular indecomposable pairs (R'',R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim}R'' > \underline{\dim}P'$, $\mathrm{top}R'' = R_{P'}(1,0) = {}^3R(1,0)$ and $\mathrm{top}R' = R_{P''}(1,0) = {}^1R(1,0)$ are $({}^3R(3t+1,0), {}^1R(6n-3t+1,0))$, where $t \in \{n,\dots,2n\}$. Thus we have

$$\begin{aligned} S_1 &= \frac{1}{\alpha_{3_{R(6n+2,0)}}} \sum_{t=n}^{2n} \alpha_{3_{R(3t+1,0)}} f_{2(t-n)}(q) q^{2n-t} \alpha_{3_{R(6n-3t+1,0)}} \\ &= (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q). \end{aligned}$$

Since $\mathrm{top}R = {}^1R(1,0)$, there are no possible regular indecomposable pairs (R'',R') in $(**)$ such that $0 \to R'' \to R \to R' \to 0$ is exact and $\mathrm{top}R' = R_{P'}(1,0) = {}^3R(1,0)$. Thus we have

$$S_2 = f_{2n}(q).$$

This means that the Ringel-Hall polynomial in this case is:

$$F^P_{RP_0} = S_1 - S_2 = (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q) - f_{2n}(q) = h_{2n+1}(q).$$

The case $\sigma_4 = \delta$

In this case $P = P(6n+3,2) = P_{\sigma_4}(+2n\delta)$, hence $\underline{\dim}R = (2n+1)\delta$. Thus R can be taken from any non-homogeneous tube.

- **R is from the tube of rank 2**

  Without loss of generality we may suppose that $R = {}^1R(4n+2,\infty)$ meaning that its regular-socle is ${}^1R(1,\infty)$ and regular-top is ${}^2R(1,\infty)$. The other case, when $R = {}^2R(4n+2,\infty)$, $P'' = n\delta + (1,2,2,1,1,1,1)$ and $R_{P'}(1,\infty) = {}^1R(1,\infty)$, $R_{P''}(1,\infty) = {}^2R(1,\infty)$, thus $\delta_{\mathrm{top}R,R_{P'}(1,\infty)} = 0$ and $\delta_{\mathrm{top}R,R_{P''}(1,\infty)} = 1$.

  The possible regular indecomposable pairs (R'',R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim}R'' > \underline{\dim}P'$, $\mathrm{top}R'' = R_{P'}(1,\infty) = {}^1R(1,\infty)$ and $\mathrm{top}R' = R_{P''}(1,\infty) = {}^2R(1,\infty)$ are $({}^1R(2t+1,\infty), {}^2R(4n-2t+1,\infty))$, where $t \in \{n,\dots,2n\}$. Thus we have

  $$\begin{aligned} S_1 &= \frac{1}{\alpha_{{}^1R(4n+2,\infty)}} \sum_{t=n}^{2n} \alpha_{{}^1R(2t+1,\infty)} f_{2(t-n)}(q) q^{2n-t} \alpha_{{}^2R(4n-2t+1,\infty)} \\ &= (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q). \end{aligned}$$

  Since $\mathrm{top}R = {}^2R(1,\infty)$, there are no possible regular indecomposable pairs (R'',R') in $(**)$ such that $0 \to R'' \to R \to R' \to 0$ is exact and $\mathrm{top}R' = R_{P'}(1,\infty) = {}^1R(1,\infty)$. Thus we have

  $$S_2 = f_{2n}(q).$$

  This means that the Ringel-Hall polynomial in this case is:

  $$F^P_{RP_0} = S_1 - S_2 = (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q) - f_{2n}(q) = h_{2n+1}(q).$$

- **R is from a tube of rank 3**

  (1) Suppose that $R = {}^1R(6n+3,0)$, meaning that its regular-socle is ${}^1R(1,0)$ and regular-top is ${}^3R(1,0)$. Also, $\underline{\dim}P' = n\delta + (0,1,1,1,0,1,0)$, $\underline{\dim}P'' = n\delta + (1,2,2,1,1,1,1)$ and $R_{P'}(1,0) = {}^2R(1,0)$, $R_{P''}(1,0) = {}^3R(1,0)$, thus $\delta_{\mathrm{top}R,R_{P'}(1,0)} = 0$ and $\delta_{\mathrm{top}R,R_{P''}(1,0)} = 1$.

  The possible regular indecomposable pairs (R'',R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim}R'' > \underline{\dim}P'$, $\mathrm{top}R'' = R_{P'}(1,0) = {}^2R(1,0)$ and $\mathrm{top}R' = R_{P''}(1,0) = {}^3R(1,0)$ are $({}^1R(3t+2,0), {}^3R(6n-3t+1,0))$, where $t \in \{n,\dots,2n\}$. Thus we have

  $$\begin{aligned} S_1 &= \frac{1}{\alpha_{{}^1R(6n+3,0)}} \sum_{t=n}^{2n} \alpha_{{}^1R(3t+2,0)} f_{2(t-n)}(q) q^{2n-t} \alpha_{{}^3R(6n-3t+1,0)} \\ &= (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q). \end{aligned}$$

Since top$R$ = ${}^3R(1,0)$, there are no possible regular indecomposable pairs (R'', R') in (∗∗) such that $0 \to R'' \to R \to R' \to 0$ is exact and top$R' = R_{P'}(1,0) = {}^2R(1,0)$. Thus we have

$$S_2 = f_{2n}(q).$$

This means that the Ringel-Hall polynomial in this case is:

$$F^P_{RP_0} = S_1 - S_2 = (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q) - f_{2n}(q) = h_{2n+1}(q).$$

(2) Suppose that $R = {}^2R(6n+3,0)$, meaning that its regular-socle is ${}^2R(1,0)$ and regular-top is ${}^1R(1,0)$. In case $F^P_{RP_0} \neq 0$, the indecomposable P must project to the regular-top of R, thus $\langle \underline{\dim} P, \underline{\dim}\,{}^1R(1,0)\rangle = 0$ must be nonzero, a contradiction. So, the Ringel-Hall polynomial in this case is $F^P_{RP_0} = 0$.

(3) Suppose that $R = {}^3R(6n+3,0)$, meaning that its regular-socle is ${}^3R(1,0)$ and regular-top is ${}^2R(1,0)$. Also, $\delta_{\text{top}R, R_{P'}(1,0)} = 1$ and $\delta_{\text{top}R, R_{P''}(1,0)} = 0$.
Since top$R$ = ${}^2R(1,0)$, there are no possible regular indecomposable pairs (R'', R') in (∗) such that $0 \to R'' \to R \to R' \to 0$ is exact and top$R' = R_{P'}(1,0) = {}^3R(1,0)$. Thus we have

$$S_1 = f_{2n+1}(q).$$

The possible regular indecomposable pairs (R'', R') in (∗∗) such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim} R'' > \underline{\dim} P''$, top$R'' = R_{P''}(1,0) = {}^3R(1,0)$ and top$R' = R_{P'}(1,0) = {}^2R(1,0)$ are $({}^3R(3t+1,0), {}^1R(6n-3t+2,0))$, where $t \in \{n+1,\dots,2n\}$. Thus we have

$$\begin{aligned} S_2 &= \frac{1}{\alpha_{{}^1R(6n+3,0)}} \sum_{t=n+1}^{2n} \alpha_{{}^3R(3t+1,0)} f_{2(t-n)-1}(q) q^{2n-t} \alpha_{{}^1R(6n-3t+2,0)} \\ &= (q-1)\sum_{s=0}^{n-1} q^s f_{2n-2s-1}(q). \end{aligned}$$

This means that the Ringel-Hall polynomial in this case is:

$$F^P_{RP_0} = S_1 - S_2 = f_{2n+1}(q) - (q-1)\sum_{s=0}^{n-1} q^s f_{2n-2s-1}(q) = h_{2n+1}(q).$$

(4) Suppose that $R = {}^1R(6n+3,1)$, meaning that its regular-socle is ${}^1R(1,1)$ and regular-top is ${}^3R(1,1)$. Also, $R_{P'}(1,1) = {}^2R(1,1)$, $R_{P''}(1,1) = {}^3R(1,1)$. By a similar argumentation as in case (1) we obtain that

$$F^P_{RP_0} = S_1 - S_2 = (q-1)\sum_{s=0}^{n} q^s f_{2n-2s}(q) - f_{2n}(q) = h_{2n+1}(q).$$

(5) Suppose that $R = {}^2R(6n+3,1)$, meaning that its regular-socle is ${}^2R(1,1)$ and regular-top is ${}^1R(1,1)$. By a similar argumentation as in case (2) we obtain that

$$F^P_{RP_0} = 0.$$

(6) Suppose that $R = {}^3R(6n+3,1)$, meaning that its regular-socle is ${}^3R(1,1)$ and regular-top is ${}^2R(1,1)$. By a similar argumentation as in case (3) we obtain that

$$F^P_{RP_0} = S_1 - S_2 = f_{2n+1}(q) - (q-1)\sum_{s=0}^{n-1} q^s f_{2n-2s-1}(q) = h_{2n+1}(q).$$

The case $\sigma_5 = \delta + \sigma_2$

In this case $P = P(6n+3,4) = P_{\sigma_5}(+2n\delta)$, thus $\underline{\dim}R = (2n+1)\delta + \underline{\dim}\,{}^1R(1,1)$, so $R = {}^1R(6n+4,1)$. Also, $\underline{\dim}P' = n\delta + (0,1,2,1,0,1,1)$, $\underline{\dim}P'' = n\delta + (1,2,2,2,1,1,0)$ and $R_{P'}(1,1) = {}^1R(1,1)$, $R_{P''}(1,1) = {}^2R(1,1)$, thus $\delta_{\mathrm{top}R, R_{P'}(1,1)} = 1$ and $\delta_{\mathrm{top}R, R_{P''}(1,1)} = 0$.

Since $\mathrm{top}R = {}^1R(1,1)$, there are no possible regular indecomposable pairs (R'', R') in $(**)$ such that $0 \to R'' \to R \to R' \to 0$ is exact and $\mathrm{top}R' = R_{P''}(1,1) = {}^2R(1,1)$. Thus we have

$$S_1 = f_{2n+1}(q).$$

The possible regular indecomposable pairs (R'', R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim}R'' > \underline{\dim}P'$, $\mathrm{top}R'' = R_{P''}(1,1) = {}^2R(1,1)$ and $\mathrm{top}R' = R_{P'}(1,1) = {}^1R(1,1)$ are $({}^1R(3t+2,1), {}^3R(6n-3t+2,1))$, where $t \in \{n+1, \dots, 2n\}$. Thus we have

$$\begin{aligned} S_2 &= \frac{1}{\alpha_{3R(6n+4,1)}} \sum_{t=n+1}^{2n} \alpha_{3R(3t+2,1)} f_{2(t-n)-1}(q) q^{2n-t+1} \alpha_{3R(6n-3t+2,1)} \\ &= (q-1)\sum_{s=0}^{n-1} q^s f_{2n-2s-1}(q). \end{aligned}$$

This means that the Ringel-Hall polynomial in this case is:

$$F^P_{RP_0} = S_1 - S_2 = f_{2n+1}(q) - (q-1)\sum_{s=0}^{n-1} q^s f_{2n-2s-1}(q) = h_{2n+1}(q).$$

The case $\sigma_6 = \delta + \sigma_3$

In this case $P = P(6n+4,4) = P_{\sigma_6}(+2n\delta)$, thus $\underline{\dim}R = (2n+1)\delta + \underline{\dim}\,{}^3R(1,0) + \underline{\dim}\,{}^1R(1,0)$, so $R = {}^3R(6n+5,0)$. Also, $\underline{\dim}P' = n\delta + (1,2,2,2,1,1,0)$, $\underline{\dim}P'' = n\delta + (1,2,3,1,1,2,1)$ and $R_{P'}(1,0) = {}^3R(1,0)$, $R_{P''}(1,0) = {}^1R(1,0)$, thus $\delta_{\mathrm{top}R, R_{P'}(1,0)} = 0$ and $\delta_{\mathrm{top}R, R_{P''}(1,0)} = 1$.

The possible regular indecomposable pairs (R'', R') in $(*)$ such that $0 \to R'' \to R \to R' \to 0$ is exact, $\underline{\dim}R'' > \underline{\dim}P'$, $\mathrm{top}R'' = R_{P'}(1,0) = {}^3R(1,0)$ and $\mathrm{top}R' = R_{P''}(1,0) = {}^1R(1,0)$ are $({}^3R(3t+1,0), {}^1R(6n-3t+4,0))$, where $t \in \{n+1, \dots, 2n+1\}$. Thus we have

$$\begin{aligned} S_1 &= \frac{1}{\alpha_{3R(6n+5,0)}} \sum_{t=n}^{2n} \alpha_{3R(3t+1,0)} f_{2(t-n)-3}(q) q^{2n-t+2} \alpha_{3R(6n-3t+4,0)} \\ &= (q-1)\sum_{s=0}^{n} q^s f_{2n-2s+1}(q). \end{aligned}$$

Since $\operatorname{top} R = {}^1R(1,0)$, there are no possible regular indecomposable pairs (R'', R') in $(**)$ such that $0 \to R'' \to R \to R' \to 0$ is exact and $\operatorname{top} R' = R_{P'}(1,0) = {}^3R(1,0)$. Thus we have

$$S_2 = f_{2n+1}(q).$$

This means that the Ringel-Hall polynomial in this case is:

$$F^P_{RP_0} = S_1 - S_2 = (q-1)\sum_{s=0}^{n} q^s f_{2n-2s+1}(q) - f_{2n+1}(q) = h_{2n+2}(q).$$

C.2 The cases $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$

In this section we present the final results of the steps from the proof of Theorem 2.10, applied to the cases $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$. A detailed description would be analogous to the one presented in the previous appendix for the case $\widetilde{\mathbb{E}}_6$.

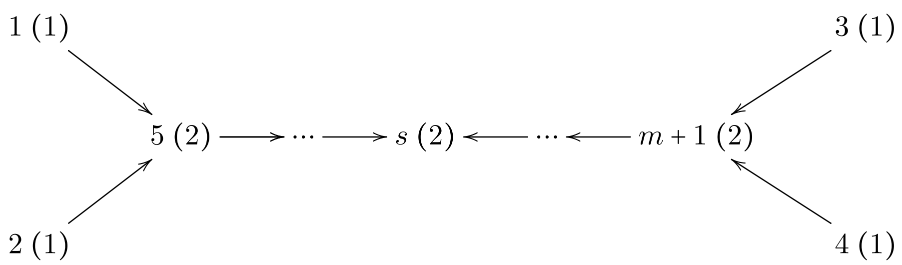
As mentioned in the proof of Theorem 2.10, we need a quiver orientation with a unique sink of defect -2. Using symmetry, one has to consider the following orientations: $Q_s^{\widetilde{\mathbb{D}}_m}$ with $s \in \{5, \dots, 5 + \lfloor \frac{m-4}{2} \rfloor\}$ the unique sink of defect -2 in the $\widetilde{\mathbb{D}}_m$ case and $Q_1^{\widetilde{\mathbb{E}}_7}$, $Q_2^{\widetilde{\mathbb{E}}_7}$ respectively $Q_1^{\widetilde{\mathbb{E}}_8}$, $Q_2^{\widetilde{\mathbb{E}}_8}$ in the cases $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$.

Over each of the quivers above we will classify the preprojectives P into families $\mathcal{P}_{\sigma+2\delta}$ or $\mathcal{P}_\sigma$ and obtain the corresponding Ringel-Hall polynomial. For simplicity, we will just list the generic form of the projectives P in each family (without mentioning σ). Each generic P in this list will be followed by the corresponding nonzero Ringel-Hall number $F^P_{RP_0}$.

Note that we will use again the notation $P = P(t,i)$, meaning that we consider the τ^{-t} shift of the projective corresponding to vertex i. Also note that in case $\underline{\dim} R$ is a multiple of δ, then we have m such nonisomorphic modules on a tube of rank m.

C.2.1 The quiver $Q_s^{\widetilde{\mathbb{D}}_m}$

Consider the quiver $Q_s^{\widetilde{\mathbb{D}}_m}$ with $s \in \{5, \dots, 5 + \lfloor \frac{m-4}{2} \rfloor\}$ being the unique sink of defect -2:



The list of the preprojectives P and the corresponding Ringel-Hall polynomials is the following:

- in case $2s < m + 6$:

  (1) $P = P((m-2)n, i)$, $F^P_{RP_0} = h_{2n}(q)$, where $i \in \{5, \dots, m+1\} \setminus \{s\}$;

  (2) $P = P((m-2)n + t, i)$, $F^P_{RP_0} = h_{2n+1}(q)$, where $t \in \{1, \dots, s-5\}$, $i \in \{4+t, m+2-t\}$;

  (3) $P = P((m-2)n + s - 4, i)$, $F^P_{RP_0} = h_{2n+1}(q)$, where $i \in \{s, \dots, m+1\}$;

  (4) $P = P((m-2)n + t, m+2-t)$, $F^P_{RP_0} = h_{2n+1}(q)$, where $t \in \{s-3, \dots, m-s+1\}$;

(5) $P = P((m-2)n + t, m - t + s - 2)$, $F^P_{RP_0} = h_{2n+2}(q)$, where $t \in \{s-3, \dots, m-s+1\}$;

(6) $P = P((m-2)n + m - s + 2, i)$, $F^P_{RP_0} = h_{2n+1}(q)$, where $i \in \{5, \dots, s\}$;

(7) $P = P((m-2)n + m - s + 2, 2s - 4)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(8) $P = P((m-2)n+t, i)$, $F^P_{RP_0} = h_{2n+2}(q)$, where $t \in \{m-s+3, \dots, m-3\}$, $i \in \{t-m+s+2, m-t+s-2\}$.

- in case $2s = m + 6$:

(1) $P = P((m-2)n, i)$, $F^P_{RP_0} = h_{2n}(q)$, where $i \in \{5, \dots, m+1\} \setminus \{s\}$;

(2) $P = P((m-2)n + t, i)$, $F^P_{RP_0} = h_{2n+1}(q)$, where $t \in \{1, \dots, s-5\}$, $i \in \{4+t, m+2-t\}$;

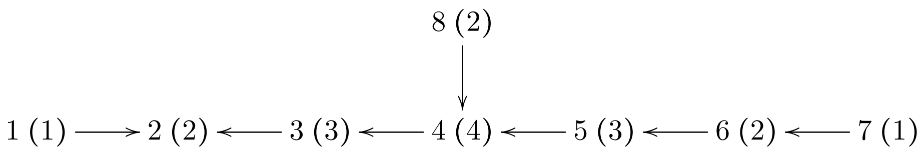
(3) $P = P((m-2)n + s - 4, i)$, $F^P_{RP_0} = h_{2n+1}(q)$, where $i \in \{5, \dots, m+1\}$;

(4) $P = P((m-2)n+t, i)$, $F^P_{RP_0} = h_{2n+2}(q)$, where $t \in \{m-s+3, \dots, m-3\}$, $i \in \{t-m+s+2, m-t+s-2\}$.

Remark C.1. In the $\widetilde{\mathbb{D}}_m$ case we have three non-homogeneous tubes with the following ranks: 2, 2, $m-2$. When $\underline{\dim} P - \underline{\dim} P_0 = \underline{\dim} R$ is a multiple of δ (this happening precisely when P is of the form $P(s-4, m-s+6)$), the Ringel-Hall polynomial will be nonzero for exactly two non-homogeneous regulars R taken from each tube. More precisely, if taken from a non-homogeneous tube $\mathcal{T}_e$ of rank m, the regular length of R is $2t_P m$ or $(2t_P+1)m$ and the regular top is $R^1_P(1,e)$ or $R^2_P(1,e)$ (see Lemma 2.2).

C.2.2 The quiver $Q_1^{\widetilde{\mathbb{E}}_7}$

Consider the quiver $Q_1^{\widetilde{\mathbb{E}}_7}$ of type $\widetilde{\mathbb{E}}_7$ having its unique sink the vertex 2, thus $P_0 = S_2$:



In this case we have three non-homogeneous tubes: $\mathcal{T}_\infty$ of rank 2, $\mathcal{T}_0$ of rank 3 and $\mathcal{T}_1$ of rank 4. The regular-simples of these tubes are:

- ${}^1R(1,\infty)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 1 & 1 & 1 & 2 & 1 & 1 & 0\end{smallmatrix}\right)$, ${}^2R(1,\infty)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 0 & 1 & 2 & 2 & 2 & 1 & 1\end{smallmatrix}\right)$;
- ${}^1R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & 0 & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & 0\end{smallmatrix}\right)$, ${}^2R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 0 & 1 & 1 & 2 & 1 & 1 & 1\end{smallmatrix}\right)$ and ${}^3R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 1 & 1 & 1 & 1 & 1 & 0 & 0\end{smallmatrix}\right)$;
- ${}^1R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 0 & 0 & 1 & 1 & 0 & 0 & 0\end{smallmatrix}\right)$, ${}^2R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & 0 & & & \\ 0 & 0 & 0 & 1 & 1 & 0 & 0\end{smallmatrix}\right)$, ${}^3R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 0 & 1 & 1 & 1 & 1 & 1 & 0\end{smallmatrix}\right)$ and ${}^4R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & 0 & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1\end{smallmatrix}\right)$.

The list of the preprojectives P and the corresponding Ringel-Hall polynomials is the following:

(1) $P = P(12n+3, 2)$, $F^P_{RP_0} = h_{2n}(q)$;

(2) $P = P(12n+4, 2)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(3) $P = P(12n+8, 2)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(4) $P = P(12n+9, 2)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(5) $P = P(12n+12, 2)$, $F^P_{RP_0} = h_{2n+2}(q)$ for $R = {}^1R(1,\infty), {}^2R(1,\infty), {}^2R(1,0), {}^3R(1,0), {}^3R(1,1), {}^4R(1,1)$;

(6) $P = P(12n, 6)$, $F^P_{RP_0} = h_{2n}(q)$;

(7) $P = P(12n+1, 6)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(8) $P = P(12n+4, 6)$, $F^P_{RP_0} = h_{2n+1}(q)$ for $R = {}^1R(1,\infty), {}^2R(1,\infty), {}^2R(1,0), {}^3R(1,0), {}^3R(1,1), {}^4R(1,1)$;

(9) $P = P(12n+7, 6)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(10) $P = P(12n+8, 6)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(11) $P = P(12n, 8)$, $F^P_{RP_0} = h_{2n}(q)$;

(12) $P = P(12n+3, 8)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(13) $P = P(12n+6, 8)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(14) $P = P(12n+9, 8)$, $F^P_{RP_0} = h_{2n+2}(q)$.

C.2.3 The quiver $Q_2^{\widetilde{\mathbb{E}}_7}$

Consider the quiver $Q_2^{\widetilde{\mathbb{E}}_7}$ of type $\widetilde{\mathbb{E}}_7$ having its unique sink the vertex 8, thus $P_0 = S_8$:

$$\begin{array}{ccccccccccccc} & & & & & & 8\,(2) & & & & & & \\ & & & & & & \uparrow & & & & & & \\ 1\,(1) & \longrightarrow & 2\,(2) & \longrightarrow & 3\,(3) & \longrightarrow & 4\,(4) & \longleftarrow & 5\,(3) & \longleftarrow & 6\,(2) & \longleftarrow & 7\,(1) \end{array}$$

In this case we have three non-homogeneous tubes: $\mathcal{T}_\infty$ of rank 2, $\mathcal{T}_0$ of rank 3 and $\mathcal{T}_1$ of rank 4. The regular-simples of these tubes are:

- ${}^1R(1,\infty)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 1 & 1 & 2 & 2 & 1 & 1 & 0 \end{smallmatrix}\right)$, ${}^2R(1,\infty)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 \end{smallmatrix}\right)$;
- ${}^1R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & 0 & & & \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}\right)$, ${}^2R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{smallmatrix}\right)$ and ${}^3R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{smallmatrix}\right)$;
- ${}^1R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}\right)$, ${}^2R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & 0 & & & \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{smallmatrix}\right)$, ${}^3R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & 1 & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}\right)$ and ${}^4R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & 0 & & & \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix}\right)$.

The list of the preprojectives P and the corresponding Ringel-Hall polynomials is the following:

(1) $P = P(12n, 2)$, $F^P_{RP_0} = h_{2n}(q)$;

(2) $P = P(12n+3, 2)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(3) $P = P(12n+6, 2)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(4) $P = P(12n+9, 2)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(5) $P = P(12n, 6)$, $F^P_{RP_0} = h_{2n}(q)$;

(6) $P = P(12n+3,6)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(7) $P = P(12n+6,6)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(8) $P = P(12n+9,6)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(9) $P = P(12n+2,8)$, $F^P_{RP_0} = h_{2n}(q)$;

(10) $P = P(12n+4,8)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(11) $P = P(12n+6,8)$, $F^P_{RP_0} = h_{2n+1}(q)$ for $R = {}^1R(1,\infty), {}^2R(1,\infty), {}^2R(1,0), {}^3R(1,0), {}^1R(1,1), {}^3R(1,1)$;

(12) $P = P(12n+8,8)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(13) $P = P(12n+10,8)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(14) $P = P(12n+12,8)$, $F^P_{RP_0} = h_{2n+2}(q)$ for $R = {}^1R(1,\infty), {}^2R(1,\infty), {}^2R(1,0), {}^3R(1,0), {}^1R(1,1), {}^3R(1,1)$.

C.2.4 The quiver $Q_1^{\widetilde{\mathbb{E}}_8}$

Consider the quiver $Q_1^{\widetilde{\mathbb{E}}_8}$ of type $\widetilde{\mathbb{E}}_8$ having its unique sink the vertex 1, thus $P_0 = S_1$:

$$\begin{array}{ccccccccccccccc} & & & & 9\,(3) & & & & & & & & & & \\ & & & & \downarrow & & & & & & & & & & \\ 1\,(2) & \longleftarrow & 2\,(4) & \longleftarrow & 3\,(6) & \longleftarrow & 4\,(5) & \longleftarrow & 5\,(4) & \longleftarrow & 6\,(3) & \longleftarrow & 7\,(2) & \longleftarrow & 8\,(1) \end{array}$$

In this case we have three non-homogeneous tubes: $\mathcal{T}_\infty$ of rank 2, $\mathcal{T}_0$ of rank 3 and $\mathcal{T}_1$ of rank 4. The regular-simples of these tubes are:

- ${}^1R(1,\infty)$ of dimension $\left(\begin{smallmatrix} & & & & & 2 & & \\ 1&2&3&2&2&1&1&0 \end{smallmatrix}\right)$, ${}^2R(1,\infty)$ of dimension $\left(\begin{smallmatrix} & & & & & 1 & & \\ 1&2&3&3&2&2&1&1 \end{smallmatrix}\right)$;
- ${}^1R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & & & 1 & & \\ 0&1&2&1&1&1&0&0 \end{smallmatrix}\right)$, ${}^2R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & & & 1 & & \\ 1&1&2&2&1&1&1&0 \end{smallmatrix}\right)$ and ${}^3R(1,0)$ of dimension $\left(\begin{smallmatrix} & & & & & 1 & & \\ 1&2&2&2&2&1&1&1 \end{smallmatrix}\right)$;
- ${}^1R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & & & 0 & & \\ 0&0&1&1&1&0&0&0 \end{smallmatrix}\right)$, ${}^2R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & & & 1 & & \\ 1&1&1&1&1&1&0&0 \end{smallmatrix}\right)$, ${}^3R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & & & 0 & & \\ 0&1&1&1&1&1&1&0 \end{smallmatrix}\right)$ and ${}^4R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & & & 1 & & \\ 1&1&2&1&1&1&1&1 \end{smallmatrix}\right)$, ${}^5R(1,1)$ of dimension $\left(\begin{smallmatrix} & & & & & 1 & & \\ 0&1&1&1&0&0&0&0 \end{smallmatrix}\right)$.

The list of the preprojectives P and the corresponding Ringel-Hall polynomials is the following:

(1) $P = P(30n+3,1)$, $F^P_{RP_0} = h_{2n}(q)$;

(2) $P = P(30n+5,1)$, $F^P_{RP_0} = h_{2n}(q)$;

(3) $P = P(30n+10,1)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(4) $P = P(30n+12,1)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(5) $P = P(30n+15,1)$, $F^P_{RP_0} = h_{2n+1}(q)$ for $R = {}^1R(1,\infty), {}^2R(1,\infty), {}^2R(1,0), {}^3R(1,0), {}^2R(1,1), {}^4R(1,1)$;

(6) $P = P(30n+18,1)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(7) $P = P(30n+20, 1)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(8) $P = P(30n+25, 1)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(9) $P = P(30n+27, 1)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(10) $P = P(30n+30, 1)$, $F^P_{RP_0} = h_{2n+2}(q)$ for $R = {}^1R(1,\infty), {}^2R(1,\infty), {}^2R(1,0), {}^3R(1,0), {}^2R(1,1), {}^4R(1,1)$;

(11) $P = P(30n, 7)$, $F^P_{RP_0} = h_{2n}(q)$;

(12) $P = P(30n+3, 7)$, $F^P_{RP_0} = h_{2n}(q)$;

(13) $P = P(30n+6, 7)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(14) $P = P(30n+9, 7)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(15) $P = P(30n+15, 7)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(16) $P = P(30n+18, 7)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(17) $P = P(30n+21, 7)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(18) $P = P(30n+24, 7)$, $F^P_{RP_0} = h_{2n+2}(q)$.

C.2.5 The quiver $Q_2^{\widetilde{\mathbb{E}}_8}$

Consider the quiver $Q_2^{\widetilde{\mathbb{E}}_8}$ of type $\widetilde{\mathbb{E}}_8$ having its unique sink the vertex 7, thus $P_0 = S_7$:

$$\begin{array}{ccccccccccccccc} & & & & 9\,(3) & & & & & & & & & & \\ & & & & \downarrow & & & & & & & & & & \\ 1\,(2) & \longrightarrow & 2\,(4) & \longrightarrow & 3\,(6) & \longrightarrow & 4\,(5) & \longrightarrow & 5\,(4) & \longrightarrow & 6\,(3) & \longrightarrow & 7\,(2) & \longleftarrow & 8\,(1) \end{array}$$

In this case we have three non-homogeneous tubes: $\mathcal{T}_\infty$ of rank 2, $\mathcal{T}_0$ of rank 3 and $\mathcal{T}_1$ of rank 4. The regular-simples of these tubes are:

- ${}^1R(1,\infty)$ of dimension $\left(\begin{smallmatrix} &&&&&1&& \\ 1&2&3&2&2&1&1&1 \end{smallmatrix}\right)$, ${}^2R(1,\infty)$ of dimension $\left(\begin{smallmatrix} &&&&&2&& \\ 1&2&3&3&2&2&1&0 \end{smallmatrix}\right)$;
- ${}^1R(1,0)$ of dimension $\left(\begin{smallmatrix} &&&&&1&& \\ 0&1&2&1&1&1&0&0 \end{smallmatrix}\right)$, ${}^2R(1,0)$ of dimension $\left(\begin{smallmatrix} &&&&&1&& \\ 1&2&2&2&2&1&1&0 \end{smallmatrix}\right)$ and ${}^3R(1,0)$ of dimension $\left(\begin{smallmatrix} &&&&&1&& \\ 1&1&2&2&1&1&1&1 \end{smallmatrix}\right)$;
- ${}^1R(1,1)$ of dimension $\left(\begin{smallmatrix} &&&&&1&& \\ 0&0&1&1&1&0&0&0 \end{smallmatrix}\right)$, ${}^2R(1,1)$ of dimension $\left(\begin{smallmatrix} &&&&&0&& \\ 0&1&1&1&0&0&0&0 \end{smallmatrix}\right)$, ${}^3R(1,1)$ of dimension $\left(\begin{smallmatrix} &&&&&1&& \\ 1&1&2&1&1&1&1&0 \end{smallmatrix}\right)$ and ${}^4R(1,1)$ of dimension $\left(\begin{smallmatrix} &&&&&1&& \\ 0&1&1&1&1&1&1&1 \end{smallmatrix}\right)$, ${}^5R(1,1)$ of dimension $\left(\begin{smallmatrix} &&&&&0&& \\ 1&1&1&1&1&1&0&0 \end{smallmatrix}\right)$.

The list of the preprojectives P and the corresponding Ringel-Hall polynomials is the following:

(1) $P = P(30n, 1)$, $F^P_{RP_0} = h_{2n}(q)$;

(2) $P = P(30n+3, 1)$, $F^P_{RP_0} = h_{2n}(q)$;

(3) $P = P(30n+6, 1)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(4) $P = P(30n+9,1)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(5) $P = P(30n+15,1)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(6) $P = P(30n+18,1)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(7) $P = P(30n+21,1)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(8) $P = P(30n+5,7)$, $F^P_{RP_0} = h_{2n}(q)$;

(9) $P = P(30n+6,7)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(10) $P = P(30n+9,7)$, $F^P_{RP_0} = h_{2n}(q)$;

(11) $P = P(30n+10,7)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(12) $P = P(30n+15,7)$, $F^P_{RP_0} = h_{2n+1}(q)$ for $R = {}^1R(1,\infty), {}^2R(1,\infty), {}^2R(1,0), {}^3R(1,0), {}^3R(1,1), {}^4R(1,1)$;

(13) $P = P(30n+20,7)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(14) $P = P(30n+21,7)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(15) $P = P(30n+24,7)$, $F^P_{RP_0} = h_{2n+1}(q)$;

(16) $P = P(30n+25,7)$, $F^P_{RP_0} = h_{2n+2}(q)$;

(17) $P = P(30n+30,7)$, $F^P_{RP_0} = h_{2n+2}(q)$ for $R = {}^1R(1,\infty), {}^2R(1,\infty), {}^2R(1,0), {}^3R(1,0), {}^3R(1,1), {}^4R(1,1)$.

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